# Absolutely ridiculous Fourier series

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Let's review just a bit about the Fourier series.

Given $f(t)$ which is $2\pi$ periodic, the Fourier series of $f(t)$ is

$$f(t) = a_0/2 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + \cdots$$

where the coefficients of $f(t)$ can be determined by integral formulas

$$\begin{aligned} a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)dt \\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)dt. \end{aligned}$$

Of course, in practice these integrals can be hard to do. For example,

$$f(t) = e^{a\cos(t)}\cos(a\sin(t))$$

is a periodic function with period $2\pi$, but I wouldn't want to do the integrals. Instead, what is often easier is to use some tricks to build new series out of old series.

Some tricks were covered in class, like superposition, differentiation, and using a familiar Fourier series like that of $sq(t)$.

This will not be about those tricks.

First, let's change things around a bit and instead write the Fourier series as

$$f(t) = \mathcal{R}e\{z_0 + z_1 e^{-it} + z_2 e^{-i2t} + z_3 e^{-i3t} + \cdots\}.$$

If you recall your prior experience with complex oscillations, you can see that this is really like the form of the series before, but with

$$\begin{aligned} z_0 &= \frac{a_0}{2} \\ z_m &= a_m + ib_m \end{aligned}$$

and the integral formulas are therefore

$$z_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$
$$z_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{imt}dt.$$

It is aesthetically irritating to have a special rule for $z_0$ and then a rule which does the rest, but it is equally irritating to have $\frac{a_0}{2}$ as the first term in the series. There are ways to make this all less irritating, but they won't buy us much more.

I'm going to drop the $\mathcal{R}e\{\}$ in what follows, but always remember to take the real part in the end.

Now we're ready to start cooking.

# 1 Fibonacci Fourier series part 1

Let's start with the Fourier series defined as

$$f(t) = F_0 + F_1 e^{-it} + F_2 e^{-i2t} + \cdots$$

where $F_0, F_1, F_2, F_3, F_4, F_5, \ldots = 1, 1, 2, 3, 5, 8, \ldots$ is the Fibonacci sequence, defined formally as

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2. \end{aligned}$$

(purists in the audience, who care about covergence, should rightly grown at this, but, we'll purify it later).

Can we find an expression for $f(t)$ and, more importantly, can we find another way to write the Fourier series for $f(t)$?

We can use the definitional formulas for the Fibonacci sequence to tell us something about the series.

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \\ F_n e^{-int} &= F_{n-1} e^{-int} + F_{n-2} e^{-int} \text{ for } n \geq 2 \\ \sum_{n \geq 2} F_n e^{-int} &= \sum_{n \geq 1} F_n e^{-i(n+1)t} + \sum_{n \geq 0} F_n e^{-i(n+2)t} \end{aligned}$$

where in the last step we adjust the indices in the sums on the two terms on the right hand side.

$$\begin{aligned} \sum_{n \geq 2} F_n e^{-int} &= e^{-it} \sum_{n \geq 1} F_n e^{-in)t} + e^{-i2t} \sum_{n \geq 0} F_n e^{-int} \\ f(t) - F_0 - F_1 e^{-it} &= e^{-it}(f(t) - F_0) + e^{-i2t} f(t) \\ f(t) - 1 - 1e^{-it} &= e^{-it}(f(t) - 1) + e^{-i2t} f(t) \\ f(t) - 1 &= e^{-it} f(t) + e^{-i2t} f(t) \\ (e^{-i2t} + e^{-it} - 1) f(t) &= -1 \\ f(t) &= \frac{-1}{e^{-i2t} + e^{-it} - 1} \\ \mathcal{R}e\{\frac{-1}{e^{-i2t} + e^{-it} - 1}\} &= \frac{\cos(2t) + \cos(t) - 1}{2\cos(2t) - 3} \end{aligned}$$

just a reminder on the last line to take real parts. Is that it? Actually, the purists who growned at the beginning are right, in that if I had worried about convergence, I would know that this result is, in some sense, nonsense. However, we can use the formulas above to re-express the series in a simpler way, which isn't nonsense.

If we think of $e^{-it}$ as a variable like $x$, the denominator is just a quadratic, $x^2 + x - 1$ which has roots $r_\pm = \frac{1}{2}(-1 \pm \sqrt{5})$ (which are ugly enough that I'll just write $r_+$ and $r_-$ for them).

$$\begin{aligned} f(t) &= \frac{-1}{(e^{-it} - r_+)(e^{-it} - r_-)} \\ &= \frac{-1}{(r_- - r_+)(e^{-it} - r_-)} + \frac{-1}{(e^{-it} - r_+)(r_+ - r_-)} \\ &= \frac{1}{\sqrt{5}(r_+ - e^{-it})} + \frac{-1}{\sqrt{5}(r_- - e^{-it})} \end{aligned}$$

where we have done partial fractions to split it up into two simpler fractions. At this point we recall something about the geometric series,

$$\frac{1}{a - x} = \frac{1}{a} \frac{1}{1 - x/a} = \frac{1}{a} \sum_{n \geq 0} (\frac{x}{a})^n = \sum_{n \geq 0} \frac{x^n}{a^{n+1}}.$$

So we can write $f(t)$ as

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\frac{e^{-int}}{r_+^{n+1}} - \frac{1}{\sqrt{5}}\sum_{n\geq 0}\frac{e^{-int}}{r_-^{n+1}} \\ &= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\frac{e^{-int}}{r_+^{n+1}} - \frac{1}{\sqrt{5}}\sum_{n\geq 0}\frac{e^{-int}}{r_-^{n+1}} \\ &= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left(\frac{2}{1+\sqrt{5}}\right)^{n+1} e^{-int} - \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left(\frac{2}{1-\sqrt{5}}\right)^{n+1} e^{-int} \end{aligned}$$

and now we see that the two things on the right hand side are just two Fourier series so

$$f(t) = \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left(\left(\frac{2}{-1+\sqrt{5}}\right)^{n+1} - \left(\frac{2}{-1-\sqrt{5}}\right)^{n+1}\right) e^{-int}$$

and we conclude that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{2}{-1+\sqrt{5}}\right)^{n+1} - \left(\frac{2}{-1-\sqrt{5}}\right)^{n+1}\right)$$

which you can check on your calculator to be true.

## 2 Purity

Unfortunately we are left wondering if $f(t)$ is really

$$f(t) = \frac{2\sin^2(t) - \cos(t)}{4\sin^2(t) + 1}$$

as the results would indicate. I believe it is the case that this is really not so. One way to check, would be to do the integral formulas and see if

$$F_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{2\sin^2(t) - \cos(t)}{4\sin^2(t) + 1}\cos(mt)dt.$$

Since this is a definite integral, you can get a calculator or computer to do this easily and you'd see that it is incorrect.

The reason for this incorrectness is that the series

$$F_0 + F_1 e^{-it} + F_2 e^{-i2t} + F_3 e^{-i3t} + \cdots$$

does not converge for any real $t$. Why doesn't it converge? Each $F_n$ is growing exponentially (like $\left(\frac{2}{-1+\sqrt{5}}\right)^{n+1}$), and the $e^{-int}$ terms are all magnitude 1, so each successive term is getting larger and larger. That is pretty non-convergent behavior.

This trick of encoding something as the Fourier coefficients of something and then using that to figure out Fourier coefficients is useful, and it would be unfortunate to abandon it.

Fortunately, there is a cure for what ails us. Instead of encoding a sequence like $F_n$ directly as the Fourier coefficients of some function, i.e. $z_m = F_m$, we can pick a number, say $p$, instead let $z_m = p^m F_m$. For a small enough $p$, the Fourier series is sure to converge and there are no worries.

## 3 Fibonacci Fourier series part 2

Leaving $p$ as a variable for now (we'll set it to $p = \frac{1}{2}$ later) let's try again with the Fourier series defined as

$$f(t) = F_0 + F_1 p e^{-it} + F_2 p^2 e^{-i2t} + F_3 p^3 e^{-i3t} \cdots.$$

We do the same tricks to the recurrence as before

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \\ F_n p^n e^{-int} &= F_{n-1} p^n e^{-int} + F_{n-2} p^n e^{-int} \text{ for } n \geq 2 \\ \sum_{n\geq 2} F_n p^n e^{-int} &= \sum_{n \geq 1} F_n p^{n+1} e^{-i(n+1)t} + \sum_{n \geq 0} F_n p^{n+2} e^{-i(n+2)t} \end{aligned}$$

and after doing similar tricks with the sum get

$$\begin{aligned} f(t) - F_0 - F_1 p e^{-it} &= p e^{-it}(f(t) - F_0) + p^2 e^{-i2t} f(t) \\ f(t) - 1 - p e^{-it} &= p e^{-it}(f(t) - 1) + p^2 e^{-i2t} f(t) \\ f(t) &= \frac{-1}{p^2 e^{-i2t} + p e^{-it} - 1} \\ \mathcal{R}e\{\frac{-1}{p^2 e^{-i2t} + p e^{-it} - 1}\} &= \frac{p^2 \cos(2t) + p\cos(t) - 1}{2p^2 \cos(2t) - 2(p^3 - p)\cos(t) - (p^4 + p^2 + 1)} \end{aligned}$$

which looks just like what we had before, if we were set $p = 1$.

With $p$ small enough that we converge (whenever $p < \frac{1+\sqrt{5}}{2}$), the above computation is valid. Taking $p = \frac{1}{2}$ we would have,

$$\begin{aligned} f(t) &= \frac{4\cos(2t) + 8\cos(t) - 16}{8\cos(2t) + 12\cos(t) - 21} \\ &= \frac{1}{2} + \frac{2\cos(t) - 11/2}{8\cos(2t) + 12\cos(t) - 21} \end{aligned}$$

and thus the following, completely obfuscated integral identity for $F_n$ for $n > 0$,

$$F_n = \frac{2^n}{\pi} \int_{-\pi}^{\pi} \frac{2\cos(t) - 11/2}{8\cos(2t) + 12\cos(t) - 21} \cos(nt) dt.$$

I have checked with numerical integration on my computer that this identity really works.