Review for 18.03

Ross A. Lippert

May 17, 2004

If you are pressed for time, skip everything and just read the cover sheet. The cover sheet contains some useful common cases.

This document is 13 pages touching upon the highlights of 14 weeks of lectures. It is terse. It is really dense. It probably still has some errors in it (if you find some, tell me and I will fix them). No warrantee is expressed or implied. There is plenty of other material to study from if you hate it.

1 The things we deal with

We started the class with real numbers and everyone knowing what they are. We added a couple more basic numerical things to this and a new way to look at functions.

1.1 Complex numbers and sinusoids

a, b, A, θ are all (related) real constants with $A \geq 0$.
Vocabulary: A is magnitude, modulus, absolute value, or amplitude. θ is phase, angle, or argument.

Anatomy of a complex number I: rectangular form: $z = a + ib$

Anatomy of a complex number II: polar form: $z = Ae^{i\theta}$

Conversion from $(A, \theta) \to (a, b)$,

$$A\cos(\theta) = a$$
$$A\sin(\theta) = b$$

use that triangle to go the other way.

Real part: $\mathcal{R}e\{a + ib\} = a$. Imaginary part: $\mathcal{I}m\{a + ib\} = b$. Real part is real. Imaginary part is **real**.

Anatomy of a sinusoid I: rectangular form

$$a\cos(\omega t) + b\sin(\omega t)$$

Anatomy of a sinusoid II: real part of complex exponential with rectangular coefficient

$$\mathcal{R}e\{(a + ib)e^{-i\omega t}\}$$

Anatomy of a sinusoid III: real part of complex exponential with polar coefficient

$$\mathcal{R}e\{Ae^{i\theta}e^{-i\omega t}\}$$

Vocabulary: here A is amplitude. θ is phase lag, or phase shift.

Anatomy of a sinusoid IV: polar form, amplitude A, phase θ

$$A\cos(\omega t - \theta)$$

Note: If you do not always take the real part or you do not always use the $e^{-i\omega t}$ exponential (not $e^{i\omega t}$), the formulas change, the warranty is void.

Example,

$$\begin{aligned}
1\cos(11t) + \sqrt{3}\sin(11t) &= \mathcal{R}e\{(1+i\sqrt{3})e^{-i11t}\} \\
&= \mathcal{R}e\{2e^{i\frac{\pi}{3}}e^{-i11t}\} \\
&= \mathcal{R}e\{2e^{-i(11t-\frac{\pi}{3})}\} \\
&= 2\mathcal{R}e\{e^{-i(11t-\frac{\pi}{3})}\} \\
&= 2\cos(11t-\frac{\pi}{3})
\end{aligned}$$

1.2 Laplace transform

The Laplace transform is a different way of looking at functions of t, whether they are complex functions, real, or vector or matrix valued. The important part of the Laplace transform is how it rearranges the information in the function so that differentials, convolutions, and other stuff are easier to work with. The basic facts of the Laplace transform are in tables. Here is a bit more.

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

If $f(t)$ is a {scalar,vector,matrix} valued function of t, then $F(s)$ is a {scalar,vector,matrix} valued function of s.

We always think of t as being real. We think of s as being complex. Often, ugly functions in t are very nice in s.

Anatomy of a Laplace transform: $F(s) = \frac{P(s)}{Q(s)}e^{-Ts}$

$P(s), Q(s)$ are polynomials. In 18.03, degree of $P(s) \leq$ degree of $Q(s)$.

Often $T = 0$. When it is not, then it comes from the t-shift rule.

Roots of $Q(s)$ are *poles*, they say something about what kinds of t-functions are in $f(t)$.

The integral in (1) does not converge unless $\mathcal{R}e\{s\}$ is greater than all the $\mathcal{R}e\{\cdot\}$'s of all the poles.

Anatomy of polynomials: They factor $Q(s) = (s-r_1)^{m_1}(s-r_2)^{m_2}\cdots(s-r_k)^{m_k}$

typically all the $m_i = 1$, no repeated roots

1.2.1 Doing Coverup

Useful fact about polynomial fractions

$$\frac{P(s)}{Q_1(s)Q_2(s)} = \frac{R_1(s)}{Q_1(s)} + \frac{R_2(s)}{Q_2(s)}$$

.

Works when

- $Q_1(s)$ and $Q_2(s)$ have no roots in common.
- degree$(P(s)) <$ degree$(Q_1(s)Q_2(s))$.

$R_i(s)$ are undetermined polynomials with degees degree$(R_i(s)) <$ degree$(Q_i(s))$. Use **coverup equations** to find them

$$\begin{aligned}
Q_1(r) = 0 \quad &\text{implies} \quad R_1(r) = \frac{P(r)}{Q_2(r)} \\
Q_2(r) = 0 \quad &\text{implies} \quad R_2(r) = \frac{P(r)}{Q_1(r)}.
\end{aligned}$$

Each root of $Q_1(s), Q_2(s)$ gives an equation for $R_2(s), R_1(s)$, respectively. If no repeated roots, you will get enough equations for all your unknowns. If roots are repeated, you will not get enough equations (use cross multiplying as a fallback strategy).

Example: Find inverse Laplace transform of

$$\frac{5}{(s-3)(s^2+4s+8)}$$

We decide to split the $s-3$ and the s^2+4s+8 parts from each other,

$$\frac{5}{(s-3)(s^2+4s+8)} \quad = \quad \frac{A}{s-3} + \frac{Bs+C}{s^2+4s+8}$$

Without coverup, you can survive by cross multiplying to get

$$1 \quad = \quad A(s^2+4s+8) + (s-3)(Bs+C)$$

then match powers of s

$$\begin{aligned} 0 &= A+B \\ 0 &= 4A-3B+C \\ 5 &= 8A-3C. \end{aligned}$$

Or use the coverup relations

$$\begin{aligned} A &= \frac{5}{3^2+4\cdot 3+8} = \frac{5}{29} \\ B(-2+2i)+C &= \frac{5}{-2+2i-3} = \frac{-25-10i}{29} \\ B(-2-2i)+C &= \frac{5}{-2-2i-3} = \frac{-25+10i}{29} \end{aligned}$$

Either way, solve for $A = \frac{5}{29}, B = -\frac{5}{29}, C = -\frac{35}{29}$.

$$\begin{aligned} \frac{5}{(s-3)(s^2+4s+8)} &= \frac{5}{29}\frac{1}{s-3} - \frac{5}{29}\frac{s+7}{s^2+4s+8} \\ &= \frac{5}{29}\frac{1}{s-3} - \frac{5}{29}\frac{(s+2)+5}{(s+2)^2+4} \\ &\to \frac{5}{29}e^{3t} - \frac{5}{29}e^{-2t}(\cos(2t) + \frac{5}{2}\sin(2t)) \end{aligned}$$

1.3 Matrices and vectors

Although more complicated versions exist, all matrices we deal with are 2×2 and all vectors are 2×1. The entries of a matrix or vector can be any kind of scalar (real or complex).

Anatomy of a vector: $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$.

1. vectors add: $\vec{u}_1 + \vec{u}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$.

    - $\vec{u}_1 + \vec{u}_2 = \vec{u}_2 + \vec{u}_1$

- $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\vec{0} + \vec{u} = \vec{u}$ (will often write 0 instead of $\vec{0}$, sorry)

2. vectors do not multiply or divide each other
3. vectors can be multiplied with scalars
   - $s \cdot \vec{u} = \vec{u} \cdot s = \begin{pmatrix} sx \\ sy \end{pmatrix}$
4. vectors can be multiplied with matrices

Anatomy of a matrix: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1. matrices add: $A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$
   - $A_1 + A_2 = A_2 + A_1$
   - $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{0} + A = A$ (will often write 0 instead of $\mathbf{0}$, sorry)
2. matrices can be multiplied with
   - scalars: $s \cdot A = A \cdot s = \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix}$
   - vectors: $A\vec{u} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$
   - other matrices: $A_1 A_2 = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$
     - **do not assume** $A_1 A_2 = A_2 A_1$
   - identity: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $I\vec{u} = \vec{u}$, $IA = A = AI$, analogous to the number 1.
3. matrices can sometimes divide
   - inverse $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (does not exist if $ad - bc = 0$)
   - $AA^{-1} = A^{-1}A = I$ like $\frac{1}{a}$ with regular scalars
   - $A\vec{u} = \vec{v}$ can be solved by $\vec{u} = A^{-1}\vec{v}$
   - not all matrices have inverses, not all division problems are solved by inverses!

     $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ has inverse, one solution $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

     $\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ has no inverse, many solutions $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

     $\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ has no inverse, no solutions

Scalars multiply vectors. Matrices multiply vectors. When is a matrix like a scalar? When it's multiplying an eigenvector,

$$A\vec{h} = \lambda\vec{h}.$$

The eigenvalue is the scalar it is acting like. Easiest way to find eigenvectors is to first find eigenvalues. They are the roots of $\det(\lambda I - A)$. Once eigenvalues are known, solve for $\vec{h}$.

General Recycling Principle: Most equations that we've seen can be rewritten with matrices and vectors instead of scalars, and they will still hold. Many concepts learned at the beginning of the course can be recycled in terms of matrices. Alternatively, if you see a matrix/vector equation and do not know what it means replace $\vec{u}(t)$ with $x(t)$ and replace every vector or matrix with some scalar, and see if that helps.

In order to make the review of equation-solving short, I'm going to use this recycling idea a lot.

2 Systems of equations

2.1 First order

First order systems are the only systems to really talk about.

Wait! What about higher order?

> The nice thing about systems of equations is that once you have decided to have more than one variable, you need only have at most one derivative in each variable.
>
> Why? Say I have a variable with two derivatives in my equations, like $\ddot{x}(t)$. Then I could add a new equation in a new variable, $y = \dot{x}(t)$, to my system (assuming the name "y" is not being used already), and everywhere there is a $\ddot{x}$, replace it with $\dot{y}$, since it is the same.
>
> Continue in this way and you add more variables to the system, but remove all but first derivatives. We call the system you are left with the *companion system.* Let's call this process *companionization.* There's an example at the end.

Anatomy of a first order system:

$$\dot{\vec{u}} = \vec{F}(\vec{u}(t), t).$$

We think of $\vec{F}(\vec{u}, t)$ as assigning a vector to each point $\vec{u}$ for each time t.

Anatomy of a first order autonymous system:

$$\dot{\vec{u}} = \vec{F}(\vec{u}(t)).$$

The vector field defined by $\vec{F}(\vec{u})$ depends only on the state, $\vec{u}$, not explicitly on t.

Anatomy of a linear constant coefficient homogeneous system:

$$\dot{\vec{u}} = A\vec{u}(t).$$

The vector field becomes just a matrix multiply. This is also autonymous.

Anatomy of a linear constant coefficient non-homogeneous system:

$$\dot{\vec{u}} = A\vec{u}(t) + \vec{v}(t).$$

A *drive* function, $\vec{q}(t)$ is added. This is also non-autonymous.

Vocab: Autonymous and homogeneous are not synonyms, but when you have constant coefficients (A does not depend on t) they pretty much are. We never have $A(t)$ so do not worry about it.

2.1.1 Solving linear constant coefficient systems

The easiest systems to tackle are the linear, constant coefficient, homogenous ones.

$$\dot{\vec{u}} = A\vec{u}$$

Guessing and superposition: guess $\vec{u}(t) = \vec{h}e^{\lambda t}$ and plug in.

$$\begin{aligned}\lambda\vec{h}e^{\lambda t} &= A\vec{h}e^{\lambda t}\\ \lambda\vec{h} &= A\vec{h}.\end{aligned}$$

Thus $\vec{h}$ is an eigenvector and λ its eigenvalue. If A is not *defective*, then you should be able to find two distinct eigenvectors. If it is, you need another kind of guess (or use the Laplace transform).

$$\begin{aligned}\vec{u}(t) &= c_1\vec{h}_1e^{\lambda_1 t} + c_2\vec{h}_2e^{\lambda_2 t}\\ \vec{u}(t) &= \Phi(t)\begin{pmatrix} c_1\\ c_2\end{pmatrix}.\end{aligned}$$

Find c_1, c_2 from initial values.

In 1 variable, $\dot{x}(t) = ax(t)$, you guess $x(t) = ce^{\lambda t}$

$$\begin{aligned}\lambda ce^{\lambda t} &= ace^{\lambda t}\\ \lambda c &= ac\\ \lambda &= a\\ x(t) &= c_1e^{at}\end{aligned}$$

The only real difference between 1-dimension and many dimensions is the use of eigenvectors. Find c_1 from initial values.

Laplace transform: transform both sides with the LT

$$\begin{aligned}s\vec{U}(s) - \vec{u}(0) &= A\vec{U}(s)\\ s\vec{U}(s) - A\vec{U}(s) &= \vec{u}(0)\\ (sI - A)\vec{U}(s) &= \vec{u}(0)\\ \vec{U}(s) &= (sI - A)^{-1}\vec{u}(0)\end{aligned}$$

inverse transform each piece of $\vec{U}(s)$ to get $\vec{u}(t)$.

In 1-d this is

$$X(s) = (s-a)^{-1}x(0)$$

and you inverse transform to get $e^{at}x(0)$.

GRP: we recycle a Laplace identity

$$\begin{aligned}e^{at} &\rightarrow \frac{1}{s-a}\\ e^{At} &\rightarrow (sI - A)^{-1}\end{aligned}$$

Note: it is worth knowing the LT trick because it always works, defective A or not.

The second easiest systems to solve are the linear, constant coefficient, non-homogeneous ones.

$$\dot{\vec{u}} = A\vec{u}(t) + \vec{v}(t)$$

Guessing and superposition: find a particular solution and the general homogeneous solution

$$\begin{aligned}\dot{\vec{u}}_p &= A\vec{u}_p + \vec{v}(t)\\ \dot{\vec{u}}_h &= A\vec{u}_h\\ \vec{u} &= \vec{u}_p + \vec{u}_h.\end{aligned}$$

From above

$$\vec{u}_h(t) \;=\; \Phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Exponential response: If $\vec{v}(t) = \vec{q}e^{rt}$, guess $\vec{u}_p(t) = \vec{h}e^{rt}$, and solve for $\vec{h}$.

$$\begin{aligned}
r\vec{h}e^{rt} &= A\vec{h}e^{rt} + \vec{q}e^{rt} \\
r\vec{h} &= A\vec{h} + \vec{q} \\
(rI - A)\vec{h} &= \vec{q} \\
\vec{h} &= (rI - A)^{-1}\vec{q} \\
\vec{u}_p(t) &= (rI - A)^{-1}\vec{q}e^{rt}
\end{aligned}$$

GRP: In 1 variable, this is the exponential response formula (ERF),

$$\begin{aligned}
\dot{x}_p &= ax_p + qe^{rt} \\
x_p(t) &= \frac{1}{r-a}qe^{rt}
\end{aligned}$$

Undetermined coefficients: More general than ERF, if $\vec{v}(t) = \vec{q}(t)e^{rt}$, and

$$\vec{q}(t) = \vec{q}_0 + \vec{q}_1 t + \vec{q}_2 t^2 + \cdots + \vec{q}_p t^p$$

is some vector polynomial in t, guess $\vec{u}_p(t) = \vec{h}(t)e^{rt}$ where

$$\vec{h}(t) = \vec{h}_0 + \vec{h}_1 t + \vec{h}_2 t^2 + \cdots + \vec{h}_p t^p$$

and solve for the $\vec{h}_i$'s. Plugging in

$$\begin{aligned}
\dot{\vec{h}}e^{rt} + r\vec{h}(t)e^{rt} &= A\vec{h}(t)e^{rt} + \vec{q}(t)e^{rt} \\
\dot{\vec{h}} + r\vec{h}(t) &= A\vec{h}(t) + \vec{q}(t) \\
\dot{\vec{h}} + r\vec{h}(t) - A\vec{h}(t) &= \vec{q}(t) \\
\vec{h}(t) &= (rI - A)^{-1}(\vec{q}(t) - \dot{\vec{h}})
\end{aligned}$$

and this gives formulas for finding the $\vec{h}$'s systematically, first solving for $\vec{h}_p$, then $\vec{h}_{p-1}$, and so on.

$$\begin{aligned}
h_p &= (rI - A)^{-1}\vec{q}_p \\
h_{p-1} &= (rI - A)^{-1}(\vec{q}_{p-1} - ph_p) \\
&\cdots \\
h_1 &= (rI - A)^{-1}(\vec{q}_1 - 2h_2) \\
h_0 &= (rI - A)^{-1}(\vec{q}_0 - h_1)
\end{aligned}$$

GRP: check that in 1 variable, this is the method of undetermined coefficients.

Note: when UC or ERF end up giving you an ∞, it means the inverses do not exist, this is *reasonance* of some sort. Things get kind of ugly in that case. Although usually a lot of work, the *Laplace transform* is powerful enough to that handle ugliness. I recommend using it, instead of memorizing more special cases.

Laplace transform: Transform both sides, including the $\vec{q}(t)$.

$$\begin{aligned}
s\vec{U}(s) - \vec{u}(0) &= A\vec{U}(s) + \vec{Q}(s) \\
(sI - A)\vec{U}(s) &= \vec{u}(0) + \vec{Q}(s) \\
\vec{U}(s) &= (sI - A)^{-1}(\vec{u}(0) + \vec{Q}(s))
\end{aligned}$$

If initial rest conditions, $\vec{u}(0) = 0$, then

$$\begin{aligned}\vec{U}(s) &= (sI - A)^{-1}\vec{Q}(s) \\ \vec{u}(t) &= e^{At} * \vec{q}(t) = \int_0^t e^{A(t-t_0)}\vec{q}(t_0)dt_0.\end{aligned}$$

Again the **GRP**. In 1-variable this is just

$$x(t) = \int_0^t e^{a(t-t_0)}q(t_0)dt_0$$

and e^{at} is the wieght function the goes with $\dot{x} - ax$. [1]

EXAMPLE: solve

$$\ddot{x} + 4\dot{x} + 8x = e^{3t}$$

with initial rest conditions $x(0) = \dot{(x)}(0) = 0$.

Companionize: there is a $\ddot{x}$ in the mix, so we invent a new variable $y = \dot{x}$ and have a new system

$$\begin{aligned} y &= \dot{x} \\ \dot{y} + 4y + 8x &= e^{3t}\end{aligned}$$

rearranging to get

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -8x - 4y + e^{3t}\end{aligned}$$

in matrices and vectors,

$$\dot{\vec{u}} = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix}\vec{u} + \begin{pmatrix} 0 \\ e^{3t}\end{pmatrix}$$

where $x(0) = y(0) = 0$.

We are left with a companion system which is first order, linear, constant coefficient, inhomogeneous.

Solution by guessing: The characteristic polynomial of A is $\lambda^2 + 4\lambda + 8$, so the eigenvalues are $-2 \pm 2i$. An eigenvector for $\lambda_1 = -2 + 2i$ is $\vec{h}_1 = \begin{pmatrix} 1 \\ -2+2i\end{pmatrix}$. An eigenvector for $\lambda_2 = -2 - 2i$ is $\vec{h}_2 = \begin{pmatrix} 1 \\ -2-2i\end{pmatrix}$. To get real solutions we take sums and differences

$$\begin{aligned}\vec{u}_h &= c_1\frac{1}{2}(\vec{h}_1 e^{\lambda_1 t} + \vec{h}_2 e^{\lambda t}) + c_2\frac{1}{2i}(\vec{h}_1 e^{\lambda_1 t} - \vec{h}_2 e^{\lambda t}) \\ &= c_1\mathcal{R}e\{\vec{h}_1 e^{\lambda_1 t}\} + c_2\mathcal{I}m\{\vec{h}_1 e^{\lambda_1 t}\} \\ &= c_1 e^{-2t}\begin{pmatrix} \cos(2t) \\ -2\cos(2t) - 2\sin(2t)\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}\sin(2t) \\ 2\cos(2t) - 2\sin(2t)\end{pmatrix}\end{aligned}$$

Now we look for $\vec{u}_p(t)$ with the ERF,

$$\begin{aligned}\vec{u}_p(t) &= (3I - A)^{-1}\begin{pmatrix} 0 \\ 1\end{pmatrix} e^{3t} \\ &= \begin{pmatrix} 3 & -1 \\ 8 & 7\end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1\end{pmatrix} e^{3t} \\ &= \frac{1}{29}\begin{pmatrix} 7 & 1 \\ -8 & 3\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} e^{3t} \\ &= \frac{1}{29}\begin{pmatrix} 1 \\ 3\end{pmatrix} e^{3t}\end{aligned}$$

---

[1] Incidently, you can see from this that all weight functions are matrix exponentials in some way.

Thus,

$$\begin{aligned}
\vec{u}(t) &= \frac{1}{29}\begin{pmatrix}1\\3\end{pmatrix}e^{3t} + c_1 e^{-2t}\begin{pmatrix}\cos(2t)\\ -2\cos(2t)-2\sin(2t)\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}\sin(2t)\\ 2\cos(2t)-2\sin(2t)\end{pmatrix}\\
x(t) &= \frac{1}{29}e^{3t} + c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)
\end{aligned}$$

and the c's need to be set by initial conditions, so there is still more to do.

Solution by Laplace:

$$\begin{aligned}
\vec{U}(s) &= (sI-A)^{-1}\vec{Q(s)}\\
&= \begin{pmatrix}s & -1\\ 8 & s+4\end{pmatrix}^{-1}\begin{pmatrix}0\\1\end{pmatrix}\frac{1}{s-3}\\
&= \frac{1}{s^2+4s+8}\begin{pmatrix}s+4 & 1\\ -8 & s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\frac{1}{s-3}\\
&= \frac{1}{(s-3)(s^2+4s+8)}\begin{pmatrix}1\\ s\end{pmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
\vec{u}(t) &= \mathcal{L}^{-1}\left[\begin{pmatrix}\frac{1}{(s-3)(s^2+4s+8)}\\ \frac{s}{(s-3)(s^2+4s+8)}\end{pmatrix}\right]\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{(s-3)(s^2+4s+8)}\right]\\
y(t) &= \mathcal{L}^{-1}\left[\frac{s}{(s-3)(s^2+4s+8)}\right].
\end{aligned}$$

Of course, if we are only after $x(t)$ we might as well skip doing the inverse transform for $y(t)$ since it gains us nothing.

$$\begin{aligned}
X(s) &= \frac{1}{(s-3)(s^2+4s+8)}\\
&= \frac{A}{s-3} + \frac{B(s+2)+C}{(s+2)^2+4}
\end{aligned}$$

and then we solve for A, B, C via coverup or cross-multiplying

$$\begin{aligned}
x(t) &= \frac{1}{(s-3)(s^2+4s+8)}\\
&= Ae^{3t} + Be^{-2t}\cos(2t) + Ce^{-2t}\sin(2t)\\
&= \frac{1}{29}e^{3t} + -\frac{1}{29}e^{-2t}(\cos(2t) + \frac{5}{2}\sin(2t))
\end{aligned}$$

2.1.2 First order Nonlinear autonymous

There is not much we can do with Nonlinear equations as such. There is usually not a great deal of structure to them. We study their behavior qualitatively in terms of *critical points*, *stability analysis*, and *graphs.*

The first order autonymous system

$$\dot{\vec{u}} = \vec{F}(\vec{u}(t)).$$

or

$$\begin{aligned}\dot{x} &= f(x(t), y(t))\\ \dot{y} &= g(x(t), y(t))\end{aligned}$$

Critical points: constant solutions, $\vec{F}(\vec{c}) = 0$ means $\vec{c}$ is a constant solution.

Linear stability analysis: if I write $\vec{u}(t) = \vec{c} + \vec{u}_1(t)$ for a critical point, $\vec{c}$, and a tiny perturbation, $\vec{u}_1(t)$, we can approximate with

$$\begin{aligned}\dot{\vec{u}}_1 &= (\vec{u}_1 \cdot \vec{\nabla})\vec{F}(\vec{c})\\ \dot{\vec{u}}_1 &= \begin{pmatrix} \partial_x f(\vec{c}) & \partial_y f(\vec{c}) \\ \partial_x g(\vec{c}) & \partial_y g(\vec{c}) \end{pmatrix} \vec{u}_1.\end{aligned}$$

The Jacobian, $\begin{pmatrix} \partial_x f(\vec{c}) & \partial_y f(\vec{c}) \\ \partial_x g(\vec{c}) & \partial_y g(\vec{c}) \end{pmatrix}$, determines the local behavior near the critical point.

Critical points are classified as nodes, spirals, saddles, or stars. When the critical point looks like a center or a degenerate, then that's a sign our approximation just is not working here (aww shucks!).

Graphs: two important kind of surfaces help with the qualitative picture, x-nullclines, where $f(x, y) = 0$, and y-nullclines, where $g(x, y) = 0$. The vector field is vertical/horizontal on x/y-nullclines. Along a nullcline the sign of the vector field is constant (left or right or up or down) until you hit a critical point (where the sign may flip or not).

Nullclines help identify large trapping regions, or *basins*.

GRP This bit about critical points in the plane and their classification is just like what we did with phase lines in one dimension. The only thing new here is that critical points can do more things, and we have these nullclines to graph.

3 Fourier series

Fourier series allow complicated periodic functions to be turned into simple periodic functions.

$$\begin{aligned}f(t) &= f(t + 2L)\\ f(t) &= \frac{1}{2}a_0 + a_1 \cos(\frac{\pi}{L}t) + b_1 \sin(\frac{\pi}{L}t) + a_2 \cos(\frac{2\pi}{L}t) + b_2 \sin(\frac{2\pi}{L}t) + a_3 \cos(\frac{3\pi}{L}t) + b_3 \sin(\frac{3\pi}{L}t) + \dots\end{aligned}$$

where coefficients are from

$$\begin{aligned}a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi}{L}t)dt\\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi}{L}t)dt.\end{aligned}$$

Or, with complex numbers:

$$f(t) = \mathcal{R}e\left\{\frac{1}{2}z_0 + z_1 e^{i\frac{\pi}{L}t} + z_2 e^{i\frac{2\pi}{L}t} + z_3 e^{i\frac{3\pi}{L}t} + \dots\right\}$$

where coefficients are from

$$z_n = a_n - b_n i = \frac{1}{L}\int_{-L}^{L} f(t)e^{-i\frac{n\pi}{L}t}dt.$$

If you use the complex form, but are asked for "the Fourier series" then take the real part and write it in terms of sin's and cos's.

In practice, watch out for an integral that does not make sense for some n (usually the z_0 integral). Do that integral right.

3.1 ERF and Fourier series

From superposition, you can get particular solutions to

$$p(D)x(t) = \mathcal{R}e\left\{\frac{1}{2}z_0 + z_1 e^{i\frac{\pi}{L}t} + z_2 e^{i\frac{2\pi}{L}t} + z_3 e^{i\frac{3\pi}{L}t} + \dots\right\}$$

by solving

$$\begin{aligned} p(D)x_0(t) &= \frac{1}{2}z_0 \\ p(D)x_1(t) &= z_1 e^{i\frac{\pi}{L}t} \\ p(D)x_2(t) &= z_2 e^{i\frac{2\pi}{L}t} \\ &\dots \end{aligned}$$

If $p(i\frac{n\pi}{L}) \neq 0$ then you use ERF,

$$x_n(t) = \frac{1}{p(i\frac{n\pi}{L})} z_n e^{i\frac{n\pi}{L}t}$$

else, $p(i\frac{n\pi}{L}) = 0$ then you use the resonant response

$$x_n(t) = \frac{t}{p'(i\frac{n\pi}{L})} z_n e^{i\frac{n\pi}{L}t}.$$

In practice, watch out for an n that ellicits the resonant response. Make sure you apply the resonant response to that n. Exponential responses are periodic. Resonant responses are not periodic!

Finally, superposition

$$x_p(t) = x_0(t) + x_1(t) + x_2(t) + \cdots$$

gets you the particular solution.

3.2 Vectors?

Yes, you could do

$$\vec{f}(t) = \mathcal{R}e\left\{\frac{1}{2}\vec{z}_0 + \vec{z}_1 e^{i\frac{\pi}{L}t} + \vec{z}_2 e^{i\frac{2\pi}{L}t} + \vec{z}_3 e^{i\frac{3\pi}{L}t} + \dots\right\}$$

where vector coefficients are from

$$\vec{z}_n = \frac{1}{L}\int_{-L}^{L} \vec{f}(t) e^{-i\frac{n\pi}{L}t} dt,$$

but that will not be on the final.

4 Special ODEs in 1-variable

Here are some things which are special about single variable equations. It is not that higher dimensional versions do not exist, but that in higher dimensions it is either way complicate or nearly useless or both.

4.1 Separability

Separable equations are nonlinear, and yet still easy.

Anatomy of a separable equation:

$$\dot{x} = f(t)g(x)$$

Solve by two integrations,

$$\begin{aligned}\int \frac{dx}{g(x)} &= \int f(t)dt + C, \\ G(x) &= F(t) + C,\end{aligned}$$

initial conditions set the constant.

- result $G(x) = F(t) + C$ is *implicit*, solve for some **continuous** $x(t)$.
- $x(t)$ may not be defined for all t, due to continuity requirement
- if $g(x_0) = 0$ there is a constant solution $x(t) = x_0$ we might miss in the integration
- the constant C could wind up anywhere in $x(t)$.
- superposition and all that is very unlikely to hold.

If you see a nonlinear equation, the first thing to check for should be separability.

Example: Solve for the general

$$\dot{x} = 2xt$$

Dividing by x separates,

$$\begin{aligned}\frac{dx}{x} &= 2tdt \\ \int \frac{dx}{x} &= \int 2tdt \\ \ln(x) &= t^2 + C \\ x(t) &= e^{t^2+C} = Ce^{t^2}\end{aligned}$$

There is a possible missing solution $x(t) = 0$, but it turns out to be present. So the general is

$$x(t) = e^{t^2+C} = C'e^{t^2}.$$

4.2 Integrating factors

First order linear with non-constant coefficients are another easy case.

Anatomy:

$$\dot{x} + p(t)x(t) = q(t)$$

Solve with an integrating factor $e^{P(t)}$ where $P(t) = \int p(t)dt$.

$$\begin{aligned}\frac{d}{dt}(e^{P(t)}x(t)) &= e^{P(t)}q(t) \\ e^{P(t)}x(t) &= \int e^{P(t)}q(t)dt + C.\end{aligned}$$

There is often a nasty integration by parts to do.

Example: Solve for the general

$$t\dot{x} + x(t) = 4t^3.$$

Standardize to

$$\dot{x} + \frac{1}{t}x(t) = 4t^2.$$

Integrating factor is $e^{ln(t)} = t$, so

$$\begin{aligned} \frac{d}{dt}(tx(t)) &= 4t^3 \\ tx(t) &= t^4 + C \\ x(t) &= t^3 + C\frac{1}{t} \end{aligned}$$

4.3 First order system nonlinear non-autonymous

The most unstructured differential equation is a first order system of nonlinear, non-autonymous equations. In this case we can not do much.

We only study these approximately and in one variable,

$$\dot{x} = F(x(t), t).$$

4.3.1 Graphing

There are a few graphical methods which are applied for drawing sketches of first order nonlinear non-autonymous systems.

Since it is not autonymous, there are probably no constant solutions.

Isoclines: Curves all with the same slope, defined by

$$m = F(x, t)$$

for some slope m. Good isoclines to sketch are typically $m = 0$ and $m = 1$ (other ones may also be important).

4.3.2 Numerical methods

Euler's method computes approximations to $x(t_i)$ for a finite set of evenly spaced t_i, starting at t_0 and ending at t_n.

The basic rule to compute $x(t_{i+1})$ from $x(t_i)$ is

$$x(t_{i+1}) = x(t_i) + (t_{i+1} - t_i)F(x(t_i), t_i)$$

It is an exercise in clerical work to carry out. It helps to keep a table around. Start with $t_0, x(t_0)$,

t	$x(t)$	$F(x,t)$	$F(x,t)\Delta t$
t_0	$x(t_0)$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
t_n	$x(t_n)$		