Complex numbers and linear equations

Anatomy	rectangular	polar	notes
complex number	$a+ib$	$Ae^{i\theta}$	a,b,θ,A real and $A\geq 0$
sinusoid	$a\cos(\omega t)+b\sin(\omega t)$	$A\cos(\omega t-\theta)$	a,b,θ,A same as above
	$\mathcal{R}e\{(a+ib)e^{-i\omega t}\}$	$\mathcal{R}e\{Ae^{i\theta}e^{-i\omega t}\}$	a,b,θ,A consistent as above

complex vocab: A is magnitude, modulus, absolute value, or amplitude. θ is phase, angle, argument.
sinusoid vocab: A is amplitude. θ is phase lag or phase shift.

Solutions	equation	solution	notes
1^{st} order LTI	$(\hat{D}-r_1\hat{I})x(t)=e^{r_2t}$	$x(t)=\frac{1}{r_2-r_1}e^{r_2t}+Ce^{r_1t}$	only if $r_1\neq r_2$
	$(\hat{D}-r\hat{I})x(t)=e^{rt}$	$x(t)=te^{rt}+Ce^{rt}$	
2^{nd} order	$p(\hat{D})x_h(t)=0$		
homogeneous	$(\hat{D}^2+b\hat{D}+k\hat{I})x_h(t)=0$		
(overdamped)	$(\hat{D}-r_1\hat{I})(\hat{D}-r_2\hat{I})x_h(t)=0$	$x_h(t)=C_1e^{r_1t}+C_2e^{r_2t}$	if $r_1\neq r_2$ real
(critical)		$x_h(t)=C_1te^{rt}+C_2e^{rt}$	if $r_1=r_2=r$
(underdamped)		$x_h(t)=e^{-at}(C_1\cos(\omega t)+C_2\sin(\omega t))$	if $r_1,r_2=-a\pm i\omega$
2^{nd} order	$p(\hat{D})x_p(t)=e^{st}$		
particular	$(\hat{D}^2+b\hat{D}+k\hat{I})x_p(t)=e^{st}$	$x_p(t)=\frac{1}{p(s)}e^{st}$	only if $p(s)\neq 0$
2^{nd} order			
general	$p(\hat{D})x_p(t)=e^{st}$	$x(t)=x_p(t)+x_h(t)$	

Linear ODE vocab: $x_p(t)$ is the particular, periodic, or (sometimes) sinusoidal, solution. $x_h(t)$ is the homogeneous, transient, autonymous, complementary, solution.

Quadratic	roots
overdamped	$r_1,r_2=-b/2\pm\sqrt{(b/2)^2-k}$
critically damped	$r_1=r_2=-b/2$
underdamped	$r_1,r_2=a\pm i\omega$
	$a=-b/2$ and $\omega=\sqrt{k-(b/2)^2}$

Generic homogeneous: If $p(\hat{D})x_h(t)=0$, and no repeated roots of $p(r)$,

$$x_h(t) \quad = \quad C_1e^{r_1t}+C_2e^{r_2t}+\ldots+C_ne^{r_nt}$$

Exponential response: If $p(\hat{D})x_p(t)=e^{st}$, and $p(s)\neq 0$,

$$x_p(t) \quad = \quad \frac{1}{p(s)}e^{st}$$

Undetermined coefficients: If $p(\hat{D})x_p(t)=q_nt^n+q_{n-1}t^{n-1}+\ldots+q_1t+q_0$, and $p(0)\neq 0$,

$$x_p(t) \quad = \quad x_nt^n+x_{n-1}t^{n-1}+\ldots+x_1t+x_0.$$

Exponential shift If $p(\hat{D})x(t)=e^{st}g(t)$,

$$p(\hat{D}-s\hat{I})(e^{-st}x(t)) \quad = \quad g(t)$$

Coverup If $Q_1(s)$ and $Q_2(s)$ have no roots in common

$$\frac{P(s)}{Q_1(s)Q_2(s)} \quad = \quad \frac{R_1(s)}{Q_1(s)}+\frac{R_2(s)}{Q_2(s)}$$

$$Q_1(r)=0 \quad \text{implies} \quad R_1(r)=\frac{P(r)}{Q_2(r)}$$

Convolution If $p(D)x(t)=f(t)$, and initial rest,

$$W(s)=\frac{1}{p(s)} \quad X(s)=W(s)F(s) \quad x(t)=\int_0^\infty w(t-p)f(p)dp$$

Systems of linear equations

Anatomy	general	specific	notes
vector	$\vec{u}$	$\begin{pmatrix} x \\ y \end{pmatrix}$	
matrix	A	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	
matrix× vector	$A\vec{u}$	$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$	
matrix× matrix	A_1A_2	$\begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{pmatrix}$	
identity matrix	$I\vec{u}=\vec{u}$ $AI=IA=A$	$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	I times anything is itself
linear system 1	$\dot{\vec{u}}=A\vec{u}$	$\dot{x}=ax(t)+by(t)$ $\dot{y}=cx(t)+dy(t)$	homogeneous version
linear system 2	$\dot{\vec{u}}=A\vec{u}+\vec{q}(t)$	$\dot{x}=ax(t)+by(t)+q_1(t)$ $\dot{y}=cx(t)+dy(t)+q_2(t)$	non-homogeneous version
trace	$\mathrm{trace}(A)$	$a+d$	$\mathrm{trace}(A)=\lambda_1+\lambda_2$
determinant	$\det(A)$	$ad-bc$	$\det(A)=\lambda_1\cdot\lambda_2$
eigenstuff	$A\vec{v}=\lambda\vec{v}$	$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=0$	λ is eigenvalue, $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, is eigenvector
characteristic polynomial	$\det(A-\lambda I)$	$\lambda^2-\mathrm{trace}(A)\lambda+\det(A)$	the roots are the eigenvalues
inverse	A^{-1}	$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$	$AA^{-1}=A^{-1}A=I$
normal mode	$\vec{v}e^{\lambda t}$	$\begin{pmatrix} v_1e^{\lambda t} \\ v_2e^{\lambda t} \end{pmatrix}$	$\vec{v}$ and λ are eigenstuff
fundamental matrix	$\Phi(t)$	$\begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}$	$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}, \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ both solutions

exponential matrix, $e^{At}=\Phi(t)\Phi(0)^{-1}$, where $\Phi(t)$ is any fundamental solutions of a homogeneous linear system

companion matrix, $\begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix}$, associated with the polynomial $\lambda^2+b\lambda+k$ or the linear operator $D^2+bD+kI$.

Common Soln's	conditions	eigenvalues	solution
saddle	$\det(A)<0$	$\lambda_1<0<\lambda_2$	$c_1\vec{v}_1e^{\lambda_1t}+c_2\vec{v}_2e^{\lambda_2t}$
unstable node	$\mathrm{trace}(A)>0, \mathrm{trace}(A)^2>4\cdot\det(A)>0$	$0<\lambda_1,\lambda_2$	$c_1\vec{v}_1e^{\lambda_1t}+c_2\vec{v}_2e^{\lambda_2t}$
stable node	$\mathrm{trace}(A)<0, \mathrm{trace}(A)^2>4\cdot\det(A)>0$	$0<\lambda_1,\lambda_2$	$c_1\vec{v}_1e^{\lambda_1t}+c_2\vec{v}_2e^{\lambda_2t}$
unstable spiral	$\mathrm{trace}(A)>0, 4\cdot\det(A)>\mathrm{trace}(A)^2>0$	$a\pm bi, a<0$	$e^{at}\mathcal{R}e\{(c_1+ic_2)\vec{v}e^{ibt}\}$
stable spiral	$\mathrm{trace}(A)<0, 4\cdot\det(A)>\mathrm{trace}(A)^2>0$	$a\pm bi, a>0$	$e^{at}\mathcal{R}e\{(c_1+ic_2)\vec{v}e^{ibt}\}$

Unommon Soln's	conditions	eigenvalues	solution
center	$\mathrm{trace}(A)=0, \det(A)>0$	$\pm bi$	$\mathcal{R}e\{(c_1+ic_2)\vec{v}e^{ibt}\}$
degenerate (unstable)	$\mathrm{trace}(A)>0, \det(A)=0$	$\lambda_1=0, \lambda_2>0$	$c_1\vec{v}_1+c_2\vec{v}_2e^{\lambda_2t}$
degenerate (stable)	$\mathrm{trace}(A)<0, \det(A)=0$	$\lambda_1=0, \lambda_2<0$	$c_1\vec{v}_1+c_2\vec{v}_2e^{\lambda_2t}$
star (stable)	$\mathrm{trace}(A)<0, \mathrm{trace}(A)^2=4\cdot\det(A)$	$\lambda_1=\lambda_2<0$	$(c_1\vec{v}_1+c_2\vec{v}_2)e^{\lambda t}$
star (unstable)	$\mathrm{trace}(A)>0, \mathrm{trace}(A)^2=4\cdot\det(A)$	$\lambda_1=\lambda_2>0$	$(c_1\vec{v}_1+c_2\vec{v}_2)e^{\lambda t}$
defective (stable)	$\mathrm{trace}(A)<0, \mathrm{trace}(A)^2=4\cdot\det(A)$	$\lambda_1=\lambda_2<0$	$(c_1\vec{v}_1+c_2(\vec{v}_1t+\vec{v}_2))e^{\lambda t}$
defective (unstable)	$\mathrm{trace}(A)>0, \mathrm{trace}(A)^2=4\cdot\det(A)$	$\lambda_1=\lambda_2>0$	$(c_1\vec{v}_1+c_2(\vec{v}_1t+\vec{v}_2))e^{\lambda t}$

For **defectives**, $\vec{v}_1$ is an eigenvector, but $\vec{v}_2$ is $(\lambda I-A)\vec{v}_2=\vec{v}_1$.

Some Laplace transforms

t-function	s-function	notes
$u(t)$	$\frac{1}{s}$	
$\delta(t)$	1	
$\dot{f}(t)$	$sF(s) - f(0)$	derivative rule
$f(t)e^{at}$	$F(s-a)$	s-shift rule
$f(t-T)u(t-T)$	$F(s)e^{-sT}$	t-shift rule
$t^n e^{at} u(t)$	$\frac{n!}{(s-a)^{n+1}}$	polynomial times exponential
$\cos(bt)e^{at}u(t)$	$\frac{s-a}{(s-a)^2+b^2}$	cosine times exponential
$\sin(bt)e^{at}u(t)$	$\frac{b}{(s-a)^2+b^2}$	sine times exponential
e^{At}	$(sI-A)^{-1}$	matrix exponential

Some matrix exponentials

A	e^{At}	notes
$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$	
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} = e^{\lambda t} I$	complete
$\begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}$	$\begin{pmatrix} e^{\lambda_1 t} & b\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \\ 0 & e^{\lambda_2 t} \end{pmatrix}$	must have $\lambda_1 \neq \lambda_2$
$\begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} e^{\lambda t} & bte^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$	defective
$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$	$\begin{pmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix}$	
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \cdot I - \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \cdot \det(A) A^{-1}$	$\lambda_1 \neq \lambda_2$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$(1 + \lambda t)e^{\lambda t} \cdot I - te^{\lambda t} \cdot \det(A) A^{-1}$	$\lambda_1 = \lambda_2 = \lambda$

Fourier series:

$$\begin{aligned} f(t) &= f(t+2L) \\ f(t) &= \frac{1}{2}a_0 + a_1 \cos(\frac{\pi}{L}t) + b_1 \sin(\frac{\pi}{L}t) + a_2 \cos(\frac{2\pi}{L}t) + b_2 \sin(\frac{2\pi}{L}t) + a_3 \cos(\frac{3\pi}{L}t) + b_3 \sin(\frac{3\pi}{L}t) + \dots \end{aligned}$$

with

$$a_n = \tfrac{1}{L} \int_{-L}^{L} f(t) \cos(\tfrac{n\pi}{L}t) dt, \quad b_n = \tfrac{1}{L} \int_{-L}^{L} f(t) \sin(\tfrac{n\pi}{L}t) dt$$