A short note on the motivation behind the defective solution for linear ODEs

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1 The short story

When we were faced with LTI homogeneous differential equations in one variable, we saw a phenomenon which happened with double roots. When we had

$$\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0$$

we could guess $x(t) = ce^{rt}$ and that guess would work when $r^2 + br + k = 0$. This usually resulted in two independent solutions since $r^2 + br + k = 0$ usually had two solutions. Thus the general was

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

However, occassionally it had only one, the *double root*. In that case, we saw that solutions of the form

$$x(t) = (c_1 + c_2 t)e^{rt}$$

would work.

In a multivariable LTI homogeneous equation,

$$\dot{\vec{u}}(t) = A\vec{u},$$

we see that solutions of the form $x(t) = \vec{\alpha}e^{\lambda t}$ will work, as long as

$$A\vec{\alpha} = \lambda\vec{\alpha}$$

which is an eigenvalue equation telling us not only what the exponential is, λ, but also what the vector part is, $\vec{\alpha}$, up to a constant multiple. Usually we had two solutions since we could find two different λ eigenvalues. Thus the general solution is

$$x(t) = c_1\vec{\alpha}_1 e^{\lambda_1 t} + c_2\vec{\alpha}_2 e^{\lambda_2 t}$$

Occassionally, we see that there is only one eigenvalue, λ. It may be that we can find two independent eigenvectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$. This is the *complete* case, and the general solution for it looks the same

$$x(t) = c_1\vec{\alpha}_1 e^{\lambda t} + c_2\vec{\alpha}_2 e^{\lambda t}.$$

But usually, a 2×2 matrix having only one eigenvalue is *defective*. We can only find one $\vec{\alpha}$ (up to a constant multiple) which goes with it.

In that case, as in the double root case for one dimensions, we must modify our guess along similar lines,

$$\vec{u}(t) = (\vec{\beta} + \vec{\alpha}t)e^{\lambda t}.$$

If we plug this guess in, we get

$$\begin{aligned}\dot{\vec{u}}(t) &= \vec{\alpha}e^{\lambda t} + \lambda(\vec{\beta} + \vec{\alpha}t)e^{\lambda t} \\ A\vec{u}(t) &= A\vec{\beta}e^{\lambda t} + A\vec{\alpha}te^{\lambda t}\end{aligned}$$

and thus

$$\vec{\alpha} + \lambda(\vec{\beta} + \vec{\alpha}t) = A\vec{\beta} + A\vec{\alpha}t.$$

If we now equate the t parts and constant parts separately, we have

$$\begin{aligned}A\vec{\alpha} &= \lambda\vec{\alpha} \\ A\vec{\beta} &= \lambda\vec{\beta} + \vec{\alpha}.\end{aligned}$$

The first equation is an eigenvalue equation for $\vec{\alpha}$ and λ. It needs to be solved first. After that is solved, and we know $\vec{\alpha}$ and λ, the second is solved for $\vec{\beta}$ by

$$(A - \lambda I)\vec{\beta} = \vec{\alpha}.$$

Combined with the already known exponential solution $x(t) = \vec{\alpha}e^{\lambda t}$, this gives the general solution for the defective case,

$$x(t) = \left(c_1\vec{\alpha} + c_2(\vec{\alpha}t + \vec{\beta})\right)e^{\lambda t}.$$

2 The more general idea

Do not read this if you are only interested in material which will be tested in this course, namely 2×2 matrices. Only read further if you have an inkling of curiousity as to what happens in more general situations with more variables.

We classified the solutions of linear constant coefficient ODE's in one variable. In the homogenous case

$$p(D)x(t) = (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)x(t) = 0$$

we found that the solutions could be related to the roots of the characteristic polynomial, $p(r)$. For every root, $p(r) = 0$ there was an exponential solution: e^{rt}.

However, when we had repeated roots, solutions did not end up looking like exponentials. They looked like polynomials times exponentials. For example,

$$\begin{aligned}(D - 2I)(D - 2I)x(t) &= 0 \\ x(t) &= C_1e^{2t} + C_2te^{2t} \\ x(t) &= (C_1 + C_2t)e^{2t}.\end{aligned}$$

In this case we have a repeated root, 2, and two solutions, one which was the usual exponential and one which was t times an exponential. You could think of the solution as a *first degree polynomial* times an exponential of the root.

Generally, when faced with repeated roots in the single variable case the answer is polynomials times exponentials where the degree of the polynomial is always the repeatedness of the root minus 1. Another example,

$$\begin{aligned}(D - 2I)^3x(t) &= 0 \\ x(t) &= (C_1 + C_2t + C_3t^2)e^{2t}.\end{aligned}$$

or to be more complicated

$$\begin{aligned}(D-2I)^3(D-5I)^2x(t) &= 0\\ x(t) &= (C_1+C_2t+C_3t^2)e^{2t}+(C_4+C_5t)e^{5t}.\end{aligned}$$

Switching to multivariable, if we have

$$D\vec{u}(t) = A\vec{u}(t)$$

instead of guessing the usual *ray* solution, which is a constant vector times an exponential, we make a more general guess, a polynomial vector times an exponential.

Guess: a d-degree polynomial vector times an exponential.

$$\vec{u}(t) = (\vec{c}_0+\vec{c}_1t+\cdots+\vec{c}_dt^d)e^{\lambda t}$$

As usual, this is not really a guess. It is something that some fairly complicated theory sits behind.

If we plug in the guess, we get something like this

$$\begin{aligned}D\vec{u}(t) &= A\vec{u}(t)\\ (\vec{c}_1+2\vec{c}_2t+\cdots+d\vec{c}_dt^{d-1})e^{\lambda t}+\lambda(\vec{c}_0+\vec{c}_1t+\cdots+\vec{c}_dt^d)e^{\lambda t} &= A(\vec{c}_0+\vec{c}_1t+\cdots+\vec{c}_dt^d)e^{\lambda t}\end{aligned}$$

we cancel the exponentials to get

$$\begin{aligned}\vec{c}_1+2\vec{c}_2t+\cdots+d\vec{c}_dt^{d-1}+\lambda(\vec{c}_0+\vec{c}_1t+\cdots+\vec{c}_dt^d) &= A(\vec{c}_0+\vec{c}_1t+\cdots+\vec{c}_dt^d)\\ \vec{c}_1+2\vec{c}_2t+\cdots+d\vec{c}_dt^{d-1}+\lambda\vec{c}_0+\lambda\vec{c}_1t+\cdots+\lambda\vec{c}_dt^d &= A\vec{c}_0+A\vec{c}_1t+\cdots+A\vec{c}_dt^d\end{aligned}$$

and, just like an undetermined coefficients problem, we set things with corresponding powers of t equal to each other, starting with the highest power first to get a chain of equations:

$$\begin{aligned}\lambda\vec{c}_d &= A\vec{c}_d\\ d\vec{c}_d+\lambda\vec{c}_{d-1} &= A\vec{c}_{d-1}\\ &\cdots\\ 2\vec{c}_2+\lambda\vec{c}_1 &= A\vec{c}_1\\ \vec{c}_1+\lambda\vec{c}_0 &= A\vec{c}_0\end{aligned}$$

That first equation doesn't look like all the others. It is an eigen-equation. You always have to solve an eigen-equation somewhere. This does two things:

- sets the value of λ to be an eigenvalue/root of $det(A-\lambda I)$
- sets the vector $\vec{c}_d$ to be the corresponding eigenvector.

We know we can set λ because we know all about polynomials. For the second part there is some complicated theory which guarantees that you can always find at least one eigenvector (up to a constant multiple) for each eigenvalue, and thus you can always get the chain started.

Once the eigen-equation for $\vec{c}_d$ and λ has been solved the rest of the equations can be re-written and solved like so:

$$\begin{aligned}(A-\lambda I)\vec{c}_{d-1} &= d\vec{c}_d\\ &\cdots\\ (A-\lambda I)\vec{c}_1 &= 2\vec{c}_2\\ (A-\lambda I)\vec{c}_0 &= \vec{c}_1\end{aligned}$$

Since λ and $\vec{c}_d$ are known, you can solve the above for $\vec{c}_{d-1}$, then you will know $\vec{c}_{d-1}$ so you can solve for $\vec{c}_{d-2}$ and so on down the chain.

"What guarantees do we have that the chain of equations can be solved?" you might wonder. Again it is a bit of complicated theory which boils down to an assurrance that this will work when you need it to, i.e. when you have a repeated eigenvalue and can't find enough independent eigenvectors to form a general solution, there will be enough independent solutions to chain equations to get you the general solution. There is no assurance that d is equal to the repeatedness of the root minus 1 (unlike in single variable LTI ODEs), so some more theory/guessing is required there.

In the 2×2 case, this is really overkill. You have only two possibilities to deal with. Either it is *complete*, and you can find two *independent* solutions which look like $\vec{c}_0 e^{\lambda t}$, where

$$A\vec{c}_0 = \lambda \vec{c}_0$$

or it is *defective*, and you can find one solution which looks like $(\vec{c}_0 + t\vec{c}_1)e^{\lambda t}$, where

$$\begin{aligned} A\vec{c}_1 &= \lambda \vec{c}_1 \\ (A - \lambda I)\vec{c}_0 &= \vec{c}_1 \end{aligned}$$