

Solution Set 6

April 16, 2008

Problem 1 (10 Points)

- (a) Let $\varepsilon > 0$ be given. We know that the natural logarithm function is continuous at 1. Thus, there exists a positive number δ such that

$$|\ln(t)| = |\ln(t) - \ln(1)| < \varepsilon, \quad (1)$$

whenever $|t-1| < \delta$. Now, if x and y are chosen from $[1, \infty)$ so that $|x-y| < \delta$, then

$$\left| \frac{x}{y} - 1 \right| < \frac{\delta}{y} \leq \delta;$$

for $y \geq 1$. Hence, from Equation (1), we get

$$|\ln(x) - \ln(y)| = \left| \ln\left(\frac{x}{y}\right) \right| < \varepsilon.$$

This shows that the logarithm function is uniformly continuous on $[1, \infty)$.

- (b) A boring start is to choose $\varepsilon > 0$. Now, if $x \neq y$ are chosen from I so that

$$|x - y| \leq \frac{\varepsilon}{K},$$

then

$$|f(x) - f(y)| = \left| \frac{f(x) - f(y)}{x - y} \right| |x - y| < K \left(\frac{\varepsilon}{K} \right).$$

(Note that the term $\left| \frac{f(x) - f(y)}{x - y} \right|$ is the slope between the points $(x, f(x))$ and $(y, f(y))$.)

- (c) The derivative of \sqrt{x} is $\frac{1}{2\sqrt{x}}$, which is evidently unbounded on $[0, 1]$ (and is undefined at 0). However, this function is uniformly continuous. The proof is as follows. Choose $\varepsilon > 0$. Now, let x and y be arbitrary numbers in $[0, 1]$ with

$$|x - y| < \varepsilon^2.$$

Thus,

$$|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}| |\sqrt{x} + \sqrt{y}| = |x - y| < \varepsilon^2.$$

This proves our assertion.

Remark:

One can conclude that \sqrt{x} is uniformly continuous, since \sqrt{x} is continuous and $[0, 1]$ is a compact interval.

Problem 2 (10 Points)

Define a polynomial $P(x) \in \mathbb{R}[x]$ by

$$P(x) := C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}.$$

Clearly, $P(0) = 0$. Also, $P(1) = 0$ by the condition on C_i 's. Therefore, P has a local extremum in $(0, 1)$, say at y . Therefore, the derivative $P'(y) = 0$, which means

$$C_0 + C_1y + C_2y^2 + \cdots + C_ny^n = 0.$$

This is what we want.

Remark:

One may argue by using the Mean Value Theorem that there exists $y \in (0, 1)$ such that $0 = P(1) - P(0) = P'(y)(1 - 0) = P'(y)$.

Problem 3 (10 Points)

- (a) We repeat the same pattern by choosing $\varepsilon > 0$. Since $f'(x) \rightarrow 0$ as $x \rightarrow \infty$, there exists $N > 0$ such that

$$x > N \Rightarrow |f'(x)| < \varepsilon.$$

Now, using the Mean Value Theorem, for each x , there exists $y(x)$ in $(x, x+1)$ such that

$$g(x) = f(x+1) - f(x) = f'(y(x))((x+1) - x) = f'(y(x)) .$$

If $x > N$, then $y(x) > N$. This shows that

$$|g(x)| = |f'(y(x))| < \varepsilon$$

whenever $x > N$. From the above argument, the result is evident.

(b) Observe that

$$f(x) := \begin{cases} +x^3, & x \geq 0, \\ -x^3, & x \leq 0. \end{cases}$$

You should not have any trouble to convince yourself that

$$f'(x) := \begin{cases} +3x^2, & x \geq 0, \\ -3x^2, & x \leq 0, \end{cases}$$

and that

$$f''(x) := 6|x| = \begin{cases} +6x, & x \geq 0, \\ -6x, & x \leq 0. \end{cases}$$

Now, if the third derivative of f exists at 0, we must have

$$\lim_{x \rightarrow 0^-} \frac{f''(x) - f''(0)}{x - 0} = f'''(0) = \lim_{x \rightarrow 0^+} \frac{f''(x) - f''(0)}{x - 0} .$$

That is

$$-6 = f'''(0) = +6,$$

a blatant contradiction.

Remark:

You cannot argue by discontinuity of f''' at 0. Discontinuity of the derivative at a point does not imply that the derivative does not exist at that point. A famous example is the following function:

$$F(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

You should check that $F'(0)$ exists and equals 0 but F' is not continuous at this point.

Problem 4 (10 Points)

Define $F(x)$ as the hint. What happens is that

$$F(a) = f(a)g(b) - f(b)g(a) = F(b).$$

The Mean Value Theorem states that there exists $c \in (a, b)$ such that

$$F(b) - F(a) = F'(c)(b - a).$$

Therefore, $F'(c) = 0$. That is,

$$0 = F'(c) = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)),$$

proving the statement.

Problem 5 (10 Points)

(a) Using the Mean Value Theorem, there exists $c \in (a, b)$ for which

$$f(b) - f(a) = f'(c)(b - a).$$

Since $f(b) - f(a) > 0$ and $b - a > 0$, this means $f'(c) > 0$.

(b) It is clear that $f(0) = 0$ and $f(1) = 1$. The Mean Value Theorem asserts that for some $c \in (0, 1)$,

$$1 = f(1) - f(0) = f'(c)(1 - 0) = f'(c).$$

Problem 6 (10 Points)

Since $a < b < c$ and $f(a) = f(b) = f(c) = 0$, we apply the Mean Value Theorem to get $p \in (a, b)$ and $q \in (b, c)$ with

$$0 = f(b) - f(a) = f'(p)(b - a)$$

and

$$0 = f(c) - f(b) = f'(q)(c - b).$$

Consequently, $p < q$ and $f'(p) = f'(q) = 0$. We again use the Mean Value Theorem to show that

$$0 = f'(q) - f'(p) = f''(s)(q - p),$$

for some $s \in (p, q) \subseteq (a, c)$.