# Solution Set 6

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## **Problem 1** (10 Points)

Let $\varepsilon > 0$ be arbitrary. Because the function $e^t$ is continuous at $t = 0$, there exist a positive real number $\delta$ such that

$$\left|e^t - 1\right| = \left|e^t - e^0\right| < \varepsilon e^{-x}$$

whenever $|t| < \delta$. Since $x_n \to x$, we can find a positive integer $N$ such that for all integer $n > N$, we get

$$|x_n - x| < \delta\,.$$

Therefore, if $n > N$, then

$$|e^{x_n} - e^x| = e^x \left|e^{x_n - x} - 1\right| < e^x \left(\varepsilon e^{-x}\right) = \varepsilon\,.$$

This proves that the exponential function is a continuous function.

## **Problem 2** (10 Points)

Let $x \geq 0$ be arbitrary. By induction on $n$, we have

$$f(x) = f\left(x^{2^n}\right)\,.$$

Substitute $x$ by $x^{\frac{1}{2^n}}$ in the above equation to yield

$$f(x) = f\left(x^{\frac{1}{2^n}}\right)\,.$$

Consequently, for any integer $n$, $f(x) = f\left(x^{2^n}\right)$.

If $x \geq 1$, then $x^{2^n} \to 1$ as $n \to -\infty$. Since $f$ is continuous,

$$\lim_{n\to-\infty} f\left(x^{2^n}\right) = f(1)\,.$$

Thus, $f(x) = f(1)$ for all $x \geq 1$.

If $0 \leq x \leq 1$, then $x^{2^n} \to 0$ as $n \to +\infty$. By continuity of $f$, we get

$$\lim_{n\to+\infty} f\left(x^{2^n}\right) = f(0)\,.$$

This shows that $f(x) = f(0)$ for each $0 \leq x \leq 1$.

Now, $f(0)$ and $f(1)$ have to be equal since $f(x)$ is continuous at $x = 1$. Hence, $f$ is a constant function.

## **Problem 3** (10 Points)

If $f(0) = 0$ or $f(1) = 1$, the result is trivial. Now, we assume that $f(0) > 0$ and $f(1) < 1$. Define $g(x) := f(x) - x$ for each $x \in I$. Thus, $g$ is a continuous function on $I$ (why?). Since $g(0) > 0$ and $g(1) < 0$, Bolzano's Theorem dictates that there exists a number $x_0 \in I$ for which $g(x_0) = 0$. Hence, $x_0$ is a fixed point of $f$ as required.

## **Problem 4** (10 Points)

If there exists an integer $k = 1, 2, \ldots, n$ such that $f\left(\frac{k}{n}\right) = f\left(\frac{k-1}{n}\right)$, then we are done. Now, suppose that $f\left(\frac{k}{n}\right) \neq f\left(\frac{k-1}{n}\right)$ for all $k = 1, 2, \ldots, n$. Let $g(x)$ denote $f(x) - f\left(x - \frac{1}{n}\right)$ for all $x \in \left[\frac{1}{n}, 1\right]$. Hence, $g$ is continuous on $\left[\frac{1}{n}, 1\right]$ (why?).

Clearly (if you are not cleared, try to expand the summation),

$$\sum_{k=1}^{n} g\left(\frac{k}{n}\right) = f(1) - f(0) = 0\,.$$

Since none of $g\left(\frac{k}{n}\right)$'s is zero, then at least one of them must be positive, and at least one must be negative. Again, we use Bolzano's Theorem and acquire a number $x_0 \in \left[\frac{1}{n}, 1\right]$ with property: $g(x_0) = 0$. Thus, $y := x_0 - \frac{1}{n} \in [0, 1]$ satisfies

$$f(y) = f\left(y + \frac{1}{n}\right)\,.$$

## **Problem 5** (10 Points)

(a) Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence in $S$, where $S := \bigcup_{k=1}^{p} I_k$ is a finite union of compact intervals $I_k$'s. Consider $T_k := \{n \in \mathbb{N} : x_n \in I_k\}$ for each $k = 1, 2, \ldots, p$. Thus,

$$\mathbb{N} := \bigcup_{k=1}^{p} T_k \,.$$

This shows that at least one of the $T_k$'s, say $T_r$, must be infinite.

Consequently, $\{x_n\}_{n\in T_r}$ is an infinite subsequence of $\{x_n\}_{n=1}^{\infty}$ that lies entirely in the compact interval $I_r$. Theorem 13.1 asserts that the subsequence $\{x_n\}_{n\in T_r}$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ in $I_r$. Therefore, $\{x_{n_i}\}_{i=1}^{\infty}$ is a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ in $S$, proving the assertion.

(b) Choose $S := \bigcup_{k=1}^{\infty} [k-1, k]$. Thus, $S = [0, \infty)$. Note that the sequence $\{n\}_{n=1}^{\infty}$ does not converge in $S$.

(c) Using the Maximum Theorem, we know that there exists $x_0 \in I$ such that $f(x_0) = \inf_{x\in I} f(x)$. Let $\delta := f(x_0)$. Then,

$$f(x) \geq \delta > 0\,,$$

as desired.

(d) A trivial example would be $f(x) := x - a$ for each $x \in (a, b)$. Clearly, $f(x) > 0$ for all $x \in (a, b)$ and $\lim_{x\to a} f(x) = 0$.

## **Problem 6** (Extra, No Points)

***Definition:*** The image of a set $S$ under a function $f$ is the set

$$f(S) := \{f(x) : x \in S\}\ .$$

The preimage of a set T under $f$ is defined by:

$$f^{-1}(T) := \{x : f(x) \in T\}\ .$$

Moreover, we normally write only $f^{-1}(a)$ to mean $f^{-1}(\{a\})$.

Basically, the problem statement is that $f$ is a real-valued function on $\mathbb{R}$ such that $\left|f^{-1}(a)\right| = 2$ for each $a \in f(\mathbb{R})$. We suppose contrary that such an $f$ exists. Without loss of generality, assume that $0 \in f(\mathbb{R})$. Let $f^{-1}(0) = \{a, b\}$, where $a < b$. Hence, either $f((a,b)) \subseteq (0, +\infty)$ or $f((a,b)) \subseteq (-\infty, 0)$. (Otherwise, there would be $c$ and $d$ in $(a, b)$ such that $f(c) < 0$ and $f(d) > 0$. Using Bolzano's Theorem, there must be $x_0$ in between $c$ and $d$ such that $f(x_0) = 0$, a contradiction.)

Without loss of generality, we assume that $f((a,b)) \subseteq (0, \infty)$. We shall prove that if $x \notin [a, b]$, then $x < 0$. Suppose contrary that $f(x) > 0$ for some $x_0 \notin [a, b]$. First assume that $x_0 < a$. Select an element $c \in (a, b)$ such that $0 < f(c) < f(x_0)$ (why is this choice of $c$ possible?). Now, let $y := \frac{f(c)}{2}$. Using Bolzano's Theorem, we can easily deduce that there exists $d_1$ and $d_2$, $a < d_1 < c$ and $c < d_2 < b$ such that

$$f(d_1) = y = f(d_2) \ .$$

Bolzano's Theorem also shows that $f(d_3) = y$ for some $d_3$, $x_0 < d_3 < a$. That is,

$$\{d_1, d_2, d_3\} \subseteq f^{-1}(y) \, ,$$

which is a contradiction. The case where $x_0 > b$ is similar.

The above argument shows that $f([a, b]) = [0, +\infty) \cap f(\mathbb{R})$. Since $[a, b]$ is a compact interval, $f|_{[a,b]}$ must attain the maximum at some point $t \in [a, b]$. According to the property of $f$, there is another point $t' \neq t$ such that $f(t') = f(t) > 0$. Therefore $t' \in [a, b]$ (using the above argument again).

If $t < t'$, then $f(x) < f(t) = f(t')$ for all $x \in (t, t')$. Suppose that the real number $r = f(x)$ for some $x \in (t, t')$. We observe that $f(x) = r$ for some $x \in [a, t)$ and for some $x \in (t', b]$, as well (why?). Hence, $\left|f^{-1}(r)\right| \geq 3$, which is a contradiction. If $t' < t$, the same contradiction arises. Thus, $t = t'$. Therefore, there is only one number (which is $t$) that yields $f(t)$. This is a contradiction again. Therefore, there is no function $f$ satisfying the conditions.