# Solution Set 4

March 23, 2008

## **Problem 1** (10 Points)

(a) Using the $n$-th Root Test,

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{2^n\sqrt{n}}\right|} = \lim_{n\to\infty} \frac{|x|}{2\sqrt[2n]{n}} = \frac{|x|}{2};$$

for $\lim_{n\to\infty} \sqrt[2n]{n} = \sqrt{\lim_{n\to\infty} \sqrt[n]{n}} = 1$. Thus, the radius of convergence is 2.

(b) Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\left(\frac{((n+1)!)^2}{(2(n+1))!} x^{n+1}\right)}{\left(\frac{(n!)^2}{(2n)!} x^n\right)} \right| = \lim_{n\to\infty} \left(\frac{n+1}{4n+2}\right)|x| = \frac{|x|}{4}.$$

Thus, the radius of convergence is 4.

(c) Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\left(\frac{x^{n+1}}{\sqrt[n+1]{n+1}}\right)}{\left(\frac{x^n}{\sqrt[n]{n}}\right)} \right| = \lim_{n\to\infty} \frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}|x| = |x|.$$

Hence, the radius of convergence is 1.

(d) Let $y := x^2$. Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{\left(\frac{(-1)^{n+1}y^{n+1}}{4^{n+1}((n+1)!)^2}\right)}{\left(\frac{(-1)^n y^n}{4^n(n!)^2}\right)}\right| = \lim_{n\to\infty}\frac{|y|}{4(n+1)^2} = 0\,.$$

Thus, the series (absolutely) converges for each real number $x$; therefore, the radius of convergence is $\infty$.

(e) Using the $n$-th Root Test,

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n+2}{n}\right)^n x^n\right|} = \lim_{n\to\infty}\left(\frac{n+2}{n}\right)|x| = |x|\,.$$

Therefore, the radius of convergence is 1.

(f) Using the $n$-th Root Test,

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{x^n}{(\ln(n))^n}\right|} = \lim_{n\to\infty}\frac{|x|}{\ln(n)} = 0\,.$$

Consequently, the radius of convergence is $\infty$.

## **Problem 2** (10 Points)

Let $x$ be a real number with $|x| < 1$. We begin with

$$f(x) := \frac{\mathrm{e}^x}{1-x} = \mathrm{e}^x\cdot\left(\frac{1}{1-x}\right) = \left(\sum_{r=0}^{\infty}\frac{x^r}{r!}\right)\left(\sum_{s=0}^{\infty}x^s\right).$$

For each $n = 0, 1, 2, \ldots$, the term $x^n$ in the Taylor expansion of $f(x)$ is given by

$$\sum_{k=0}^{n}\left(\frac{x^k}{k!}\right)\left(x^{n-k}\right) = \left(\sum_{k=0}^{n}\frac{1}{k!}\right)x^n\,.$$

Hence, $c_n = \sum\limits_{k=0}^{n}\frac{1}{k!}$.

## **Problem 3** (10 Points)

(a) i. If $f$ is even, then

$$f \circ g(-x) = f(g(-x)) \in \{f(+g(x)), f(-g(x))\} = \{f(g(x))\} .$$

Hence, $f \circ g(-x) = f \circ g(x)$. Therefore, $f \circ g$ is even.

ii. If $f$ is odd and $g$ is even, then

$$f \circ g(-x) = f(g(-x)) = f(g(x)) = f \circ g(x) .$$

Consequently, $f \circ g$ is even.

iii. If both $f$ and $g$ are odd, we get

$$f \circ g(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -f \circ g(x) .$$

Thus, $f \circ g$ is odd.

(b) Let $x$ and $y$ be arbitrary real numbers such that $x < y$. Since $g$ is a decreasing function,

$$g(x) \geq g(y) .$$

Since $f$ is also decreasing,

$$f(g(x)) \leq f(g(y)) .$$

Thus, $f \circ g(x) \leq f \circ g(y)$, meaning that $f \circ g$ is increasing.

(c) The condition for a function $f$ to be an injection is that, for all $x$ and $y$ (in the domain of $f$), $f(x) = f(y)$ implies $x = y$. This is equivalent to the contrapositive: for all $x$ and $y$, $x \neq y$ implies $f(x) \neq f(y)$.

Now, suppose that $f$ is strictly increasing on $\mathbb{R}$. For each pair of real numbers $x$ and $y$, if $x \neq y$, we must have $x < y$ or $x > y$. Without loss of generality, we may assume that $x < y$. Hence, $f(x) < f(y)$ as $f$ is strictly increasing. Thus, $f(x) \neq f(y)$ for all $x \neq y$. This proves that $f$ is injective.

## Problem 4 (10 Points)

(a) Let $A \subseteq \mathbb{R}$. Define $f|_A$ to be the restriction of $f$ on $A$. To gain full credit, one must mention (by writing or drawing a graph) that the function $f|_{[0,+\infty)}$ is strictly increasing (and hence, one-to-one). Hence, in this domain, $f$ has a unique inverse function $g$. To compute $g$, we note that if $x = g(t)$, then

$$t = f(g(t)) = f(x) = x^2 + 2x + 2\,.$$

Consequently, $g(t) = x = -1 \pm \sqrt{t-1}$. Note that $x \in [0, +\infty)$. This implies $g(t) = -1 + \sqrt{t-1}$, defined for $t \in [2, \infty)$.

(b) By plotting graph, we see that $f|_{[-2,0]}$ is not injective. (We can also check that $f(0) = 2 = f(-2)$.) Thus, $f|_{[-2,0]}$ does not have an inverse.

(c) Since $f|_{(-\infty,-2]}$ is strictly decreasing, it is an injective. Let $h$ be its inverse. Similar to (a), if $x = h(t)$, then

$$t = f(h(t)) = f(x) = x^2 + 2x + 2\,.$$

This gives $h(t) = x = -1 \pm \sqrt{t-1}$. As $x \in (-\infty, -2]$, we must have $h(t) = -1 - \sqrt{t-1}$. Clearly, the domain of $h$ is $[2, \infty)$.

## Problem 5 (10 Points)

(a) Let $\mathcal{D}_f \subseteq \mathbb{R}$ be the natural domain of $f$. Hence, $x \in \mathcal{D}_f$ **if and only if** all of the following conditions are satisfied:

i. $9 - \sqrt{25 - \sqrt{x}} \geq 0$,

ii. $25 - \sqrt{x} \geq 0$, and

iii. $x \geq 0$.

The second and the third conditions are satisfies iff $0 \leq x \leq 625$. However, if $x \in [0, 625]$, then $x$ automatically works for the first conditions (hence, it is inferior and we may disregard it). This argument shows that $\mathcal{D}_f = [0, 625]$

*Remark:*

Most students did not write that $x \in [0, 625]$ if and only if $x \in \mathcal{D}_f$. Nearly all of these students arrived at proving that $\mathcal{D}_f \subseteq [0, 625]$ without showing (or at least mentioning the reason) that they are equal.

(b) Suppose $0 \leq x < y \leq 625$. Then, $\sqrt{x} < \sqrt{y}$. Hence,

$$25 - \sqrt{x} > 25 - \sqrt{y} .$$

Therefore, $\sqrt{25 - \sqrt{x}} > \sqrt{25 - \sqrt{y}}$ and

$$9 - \sqrt{25 - \sqrt{x}} < 9 - \sqrt{25 - \sqrt{y}} .$$

This shows that $f(x) < f(y)$. Hence, $f$ is strictly increasing.

## **Problem 6** (10 Points)

(a) Assume that the functional equation $f(x) = \frac{1}{f(x)}$ is satisfied. Thus,

$$(f(x))^2 = 1 .$$

Consequently, $f(x) = \pm 1$ for all $x \in \mathbb{R}$. It is easy to see that this is already a full description of all functions $f$ with this property.

*Remark:*

Many people gave extra incorrect information about $f$ and hence, they got reduction as rewards.

(b) Suppose $x$ and $y$ belong to the domain of $f$ (which may not be the whole $\mathbb{R}$). Suppose that $x < y$. Let $p > 0$ is a period of $f$. Our task is to show that $f(x) = f(y)$. (This will prove that $f$ is a constant function. Why?)

Choose a positive integer $N$ sufficiently large that $Np > y - x$. Therefore, $f(x) = f(x + Np)$ (why?). However, as $f$ is increasing, we must have

$$f(x) \leq f(y) \leq f(x + Np) = f(x) ,$$

since $x < y < x + Np$. This proves that $f(x) = f(y)$, as desired.

Note that tan does not fall in this category because it is NOT increasing on its domain ($\tan(0) = 0 > -1 = \tan\left(\frac{5\pi}{4}\right)$ but $0 < \frac{5\pi}{4}$). (Some people wrote that it was because tan is not defined for the some real numbers. This is, nevertheless, NOT the case.)

(c) Such a function is given by

$$f(x) = \begin{cases} +1\,, & \text{if } x \text{ is rational,} \\ -1\,, & \text{otherwise.} \end{cases}$$

In this case, $f(x) = f(x + p)$ for all rational $p$. Therefore, $f$ has no minimal period. (The graph of $f$ is impossible to draw. It fluctuates between $+1$ and $-1$ rapidly.)

*Remark:*

Can you show that the only continuous functions on $\mathbb{R}$ with no minimal period are constant functions?