18.100A – PROBLEM SET #2
SOLUTIONS

Problem 1 (10 Points)**.** First, we notice

$$\sqrt{n+1}-\sqrt{n}=\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}=\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

For $a_n=\sqrt{n}$ we therefore obtain

$$|a_{n+1}-a_n|=\frac{1}{\sqrt{n+1}+\sqrt{n}}<\frac{1}{2\sqrt{n}}.$$

Thus, for given $\epsilon>0$, we can choose $N>1/(4\epsilon^2)$ so that

$$|a_{n+1}-a_n|<\epsilon, \quad \text{for all } n\geq N,$$

as desired.

Problem 2 (10 Points)**.**

(a) Since $\{x_n\}$ takes only finitely many values $a_1,\dots,a_k$, we see that

$$B\leq x_n\leq C, \quad \text{for all } n,$$

where $B=\min\{a_1,\dots,a_k\}$ and $C=\max\{a_1,\dots,a_k\}$ (i.e., the numbers B and C are the smallest and largest value among the $a_1,\dots,a_k$, respectively.). Hence $\{x_n\}$ is a bounded sequence and, by the Bolzano-Weierstrass Theorem (Theorem 6.3), the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with limit K, say. By the cluster point theorem (Theorem 6.2), K must be a cluster point of the whole sequence $\{x_n\}$.

(b) (i) Since $|\cos x|\leq 1$ for all $x\in\mathbb{R}$, we conclude that $|b_n|\leq 1$ for all n. Therefore $\{b_n\}$ is always a bounded sequence and, by the Bolzano-Weierstrass Theorem (Theorem 6.3), the sequence $\{b_n\}$ has to have a convergent subsequence.

(ii) The sequence $b_n=a_n/(1+a_n)$ is unbounded if a_n converges to -1. Hence the Bolzano-Weierstrass theorem is of no use in this case. Moreover, the following example shows that b_n can have in fact no convergent subsequence: Take $a_n=n/(1-n)$. (Note that $a_n\neq-1$). Then we have $b_n=n$ (by simple algebra) and it is easy to see that $b_n=n$ cannot have a convergent subsequence. (For the sake of completeness, here is the argument. Suppose that $\{b_{n_k}\}$ is a convergent subsequence of $b_n=n$. Since convergence of a sequence implies its boundedness, we have that $b_{n_k}\leq C$ for all k and some constant C. However, we have that $b_{n_k}=n_k$ tends to ∞ as $k\to\infty$, contradicting the bound from above given by C.)

- Since $1/(1+|x|)\leq 1$ for all $x\in\mathbb{R}$, we have that $|b_n|\leq 1$ for all n. Therefore $\{b_n\}$ is bounded and has a convergent subsequence, thanks to the Bolzano-Weierstrass theorem.

Problem 3 (10 Points)**.** We claim that $\sup S$ is a lower bound for the set T, i. e.,

$$\sup S \leq t, \quad \text{for all } t \in T.$$

To prove this, we argue by contradiction as follows. Suppose the statement above was false. Then there exist some $t_0 \in T$, say, such that

$$t_0 < \sup S.$$

Since $t \geq s$ for all $s \in S$ and $t \in T$, we deduce that $t_0 \geq s$ for all s. Hence t_0 is also an upper bound for S, but by the previous inequality we have that $t_0 < \sup S$. Hence t_0 is an upper bound for S that is strictly smaller than $\sup S$, which contradicts the fact that $\sup S$ is the least upper bound for S.

Next, we claim that $\sup S$ being a lower bound for T implies that

$$\sup S \leq \inf T.$$

Indeed, we see this again by a proof by contradiction. Suppose, on the contrary, that $\sup S > \inf T$ holds. Since $\sup S$ is a lower bound for T, this contradicts the fact that $\inf T$ is the greatest lower bound for T. For otherwise $\sup S$ would be a strictly greater lower bound for T. Hence $\sup S > \inf T$ cannot be true and, consequently, we find that $\sup S \leq \inf T$.

Problem 4 (10 Points)**.**

(a) In order to complete the argument, one has to show that $\lim_{k\to\infty} |\sum_{n=1}^{k} a_n| = |\lim_{k\to\infty} \sum_{n=1}^{k} a_n|$ holds. Your critique was supposed to point out that "by taking the limit $k \to \infty$" has to be detailed.

(b) The proof of the usual (and extended) triangle inequality (see Section 2.4) shows that

$$-|a_1| - \ldots - |a_n| \leq a_1 + \cdots + a_n \leq |a_1| + \cdots + |a_n|$$

for all $n \geq 1$. Let $\{s_n\}$ and $\{t_n\}$ denote the sequences of partial sums for $\sum a_n$ and $\sum |a_n|$ respectively. Then the inequality above gives us

$$-t_n \leq s_n \leq t_n, \quad \text{for all } n \geq 1.$$

Since $\lim_{n\to\infty} t_n = \sum_{n=1}^{\infty} |a_n|$ and $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$. This shows that

$$-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|,$$

which is equivalent to the desired statement $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$.

Problem 5 (10 Points)**.**

(a) We have **convergence**. To see this, note that

$$0 \leq \frac{\sqrt{n}}{n^2+1} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}, \quad \text{for } n \geq 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, as one easily concludes from the integral test; see Example 7.5A in Section 7.5. By the comparison test, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges as well.

(b) We have **convergence.** To show this, we use the ratio test. For $a_n = \frac{n^2}{2^n}$, we find

$$|\frac{a_{n+1}}{a_n}| = \frac{(n+1)^2}{2^{n+1}}\frac{2^n}{n^2} = \frac{1}{2}\frac{(n+1)^2}{n^2} = \frac{1}{2}\frac{n^2+2n+1}{n^2} \to L = \frac{1}{2}.$$

By the ratio test, the fact that $L < 1$ implies that $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

(c) We have **convergence.** Since $|\cos n| \leq 1$, we have

$$0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}, \quad \text{for all } n \geq 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, thanks to the integral test. Hence, by the comparison test, the series $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ also converges. Thus $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges too.

(d) We have **convergence**. We use the ratio test. For $a_n = \frac{(n!)^2}{(2n)!}$, we find

$$|\frac{a_{n+1}}{a_n}| = \frac{((n+1)!)^2}{(2(n+1))!}\frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n^2+2n+1}{4n^2+6n+2} \to L = \frac{1}{4}.$$

Since $L < 1$, the ratio test implies convergence.

(e) We have **convergence.** To see this, we use the n-th root rest. For $a_n = (n+1)^n/(2n+1)^n$, we obtain

$$\sqrt[n]{|a_n|} = \frac{n+1}{2n+1} \to L = \frac{1}{2}.$$

Since $L < 1$, we have convergence.

(f) We have **divergence.** To prove this, we use the integral test, which implies that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges if the integral

$$\int_2^{\infty} \frac{dx}{x \ln x}$$

diverges. Indeed, by defining $y = \ln x$ so that $dy = \frac{dx}{x}$, we find

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{dy}{y} = \ln y \Big|_{y=\ln 2}^{\infty} = \infty.$$

(g) We have **divergence.** First, we recall the inequality $\sin x \geq \frac{1}{2}x$ for $0 \leq x \leq 1$. Since $0 \leq 1/n \leq 1$ for all $n \geq 1$, this inequality shows that

$$\sin\left(\frac{1}{n}\right) \geq \frac{1}{2n}, \quad \text{for all } n \geq 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, we see that $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges by the comparison test.

(h) We have **convergence.** This is a nice one and follows from the ratio test. Let $a_n = \frac{2^n n!}{n^n}$. Then

$$\begin{aligned} |\frac{a_{n+1}}{a_n}| &= \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\frac{n^n}{2^n n!} = 2(n+1)\frac{n^n}{(n+1)^{n+1}} \\ &= 2\frac{n^n}{(n+1)^n} = 2\frac{1}{(1+1/n)^n} \to L = \frac{2}{e}. \end{aligned}$$

Since $e = 2.71... > 2$, we have $L < 1$ and the series converges thanks to the ratio test.

Problem 6 (Extra Problem). This is an extra (no credit) problem. Solution will be posted soon.