

18.100A – PROBLEM SET #2
SOLUTIONS

Problem 1 (10 Points).

(4 Points) Suppose that $L := \lim_{n \rightarrow \infty} a_n$ exists. Using that $a_{n+1} = 2a_n^2$, we obtain (using the product rule of the algebraic limit theorems, see Theorem 5.1)

$$L = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} 2a_n^2 = 2 \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} a_n \right) = 2L^2.$$

Hence, $L(2L - 1) = 2L^2 - L = 0$. Consequently, $L = 0$ or $\frac{1}{2}$.

(6 Points) We shall proceed by induction on n that $a_n = \frac{1}{2} (2a_0)^{2^n}$. For $n = 0$, this is trivial. Assume that $a_k = \frac{1}{2} (2a_0)^{2^k}$ for some integer $k \geq 0$. Therefore,

$$a_{k+1} = 2a_k^2 = 2 \left(\frac{1}{2} (2a_0)^{2^k} \right)^2 = \frac{1}{2} \left((2a_0)^{2^k} \right)^2 = \frac{1}{2} (2a_0)^{2^{k+1}}.$$

This completes the induction. Hence, $a_n = \frac{1}{2} (2a_0)^{2^n}$ for all nonnegative integer n .

Clearly, if $0 < a_0 < \frac{1}{2}$, then $0 < 2a_0 < 1$ and thus,

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \left(\lim_{n \rightarrow \infty} (2a_0)^{2^n} \right) = \frac{1}{2} \left(\lim_{N \rightarrow \infty} (2a_0)^N \right) = 0,$$

using that $\alpha^n \rightarrow 0$ as $n \rightarrow \infty$ if $|\alpha| < 1$; see Theorem 3.4.

On the other hand, if $a_0 > \frac{1}{2}$, then $2a_0 > 1$. We then get

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \left(\lim_{n \rightarrow \infty} (2a_0)^{2^n} \right) = \frac{1}{2} \left(\lim_{N \rightarrow \infty} (2a_0)^N \right) = \infty,$$

using that $\{\alpha^n\}$ tends to ∞ if $\alpha > 1$, by Theorem 3.4 again.

The special case is when $a_0 = \frac{1}{2}$, in which case the sequence $\{a_n\}$ is constant, with all terms being $\frac{1}{2}$.

Problem 2 (10 Points).

(7 Points) Define $h_n := a^{\frac{1}{n}} - 1$. Since $a > 1$, then $h_n > 0$. Now, according to the Binomial Theorem (see proof of Theorem 3.4), we have

$$a = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2!} h_n^2 + \cdots + h_n^n > 1 + nh_n.$$

(Alternatively, you could use Bernoulli's inequality to deduce that $(1 + h_n)^n > 1 + nh_n$ holds.) Thus, $0 < h_n < \frac{a-1}{n}$. That is, h_n can be arbitrarily small (if n is sufficiently large). Therefore, $\lim_{n \rightarrow \infty} h_n = 0$, proving our assertion.

(3 Points) Note that if $0 < a < 1$, then $1/a > 1$. Consider the sequence $b_n = (1/a)^{1/n}$. From part (a) we conclude $\lim b_n = 1$. Furthermore, we have $a^{1/n} = 1/b_n$ (by simple algebra). Finally, by the algebraic limit theorem (Theorem 5.1), we infer from $\lim b_n = 1$ that $\lim a^{1/n} = \lim(1/b_n) = 1/\lim b_n = 1/1 = 1$.

Problem 3 (10 Points). For $|r| \leq 1$, we see that $|a_n| \leq \frac{1}{n!}$, which implies that $a_n \rightarrow 0$ as $n \rightarrow \infty$. Now we assume that $|r| > 1$. Let N be a positive integer such that $N > 2|r|$. Thus, for each $n \geq N$,

$$\begin{aligned} |a_{n+1}| &= \frac{|r|^{n+1}}{(n+1)!} = \frac{|r|}{n+1} \left(\frac{|r|^n}{n!} \right) \\ &< \frac{|r|}{n} \left(\frac{|r|^n}{n!} \right) \leq \frac{|r|}{N} \left(\frac{|r|^n}{n!} \right) \\ &< \frac{1}{2} \left(\frac{|r|^n}{n!} \right) = \frac{1}{2} |a_n|. \end{aligned}$$

Thus, if $n \geq N$, then $|a_{n+1}| > \frac{1}{2} |a_n|$ and therefore,

$$|a_n| < \frac{|a_N|}{2^{n-N}}, \quad \text{for } n \geq N,$$

(this can be proven by induction on n and we shall omit it). This shows that

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Indeed, the inequality derived above gives us

$$0 \leq |a_n| < K \left(\frac{1}{2} \right)^n, \quad \text{for } n \geq N,$$

with the constant $K = |a_N|2^N$. By Theorem 3.4, we have $\lim_n K \frac{1}{2^n} = K \lim_n \frac{1}{2^n} = 0$. So, by the Squeeze Theorem 5.2, we conclude that $\lim a_n = 0$ also.

Problem 4 (10 Points).

(3 Points) If L is the limit of the sequence $\{a_n\}$, then

$$L = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \cos(a_n) = \cos \left(\lim_{n \rightarrow \infty} a_n \right) = \cos(L).$$

The numerical value of L is 0.7390851332.

(7 Points) Assume that $a_n \rightarrow L$ and define $e_n = a_n - L$.

First note that $|a_1| \leq 1$. Thus, $\cos(1) \leq a_2 \leq 1$. As a result,

$$\cos(1) \leq a_3 \leq \cos(\cos(1)) < 1.$$

By induction on n , we can show that for all $n > 3$, $\cos(1) < a_n < 1$. This shows that for all $n > 3$,

$$0 < \frac{a_n + L}{2} < \frac{1 + L}{2} < \frac{\pi}{2},$$

as $0 < L < \frac{\pi}{2}$ and $0 < a_n < 1$. Set $K := \sin\left(\frac{1+L}{2}\right)$; hence K is a constant with $0 < K < 1$.

Note that

$$|e_{n+1}| = |a_{n+1} - L| = |\cos(a_n) - \cos(L)|.$$

Using the trigonometric identity $\cos(x) - \cos(y) = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$, we obtain

$$|e_{n+1}| = 2 \left| \sin\left(\frac{a_n + L}{2}\right) \right| \left| \sin\left(\frac{a_n - L}{2}\right) \right|.$$

It is known that $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$. Also, $e_n = a_n - L$. Consequently,

$$|e_{n+1}| \leq 2 \left| \sin \left(\frac{a_n + L}{2} \right) \right| \left| \frac{e_n}{2} \right| = \left| \sin \left(\frac{a_n + L}{2} \right) \right| |e_n|.$$

Therefore, for each $n > 3$,

$$|e_{n+1}| \leq K |e_n|;$$

for $0 < \sin \left(\frac{a_n + L}{2} \right) < \sin \left(\frac{1+L}{2} \right) = K$, as desired.

Problem 5 (10 Points). Let ε be any positive real number. Since $\frac{a_n}{b_n} \rightarrow L$ as $n \rightarrow \infty$, then $\left| \frac{a_n}{b_n} \right| \rightarrow |L|$ as $n \rightarrow \infty$. Thus, there exists a positive integer N_1 such that

$$\left| \left| \frac{a_n}{b_n} \right| - |L| \right| < |L| + 1,$$

whenever $n > N_1$. Hence,

$$(1) \quad \left| \frac{a_n}{b_n} \right| < (|L| + 1) + |L| = 2|L| + 1.$$

Now, since $b_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, there exists a positive integer N_2 for which

$$(2) \quad |b_n| < \left(\frac{1}{2|L| + 1} \right) \varepsilon,$$

for all $n > N_2$. Let $N := \max\{N_1, N_2\}$; according to Equation (1) and Equation (2), if $n > N$, then

$$|a_n| < (2|L| + 1) |b_n| < \varepsilon.$$

Thus, $\lim_{n \rightarrow \infty} a_n = 0$, as desired.

Problem 6 (10 Points). Let $\{a_n\}$ be a sequence, and suppose you would like to show $\lim_n a_n = L$.

If you can prove that $|a_n - L| < \epsilon$ for $n \gg 1$ with $0 < \epsilon < 1$, say, then this statement is automatically true for all larger $\epsilon' \geq 1$, since

$$|a_n - L| < \epsilon \quad \Rightarrow \quad |a_n - L| < \epsilon',$$

whenever $\epsilon' \geq \epsilon$.

Problem 7 (10 Points).

(3 Points) Suppose $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L \in [0, 1)$. Then choose $\epsilon = (1 - L)/2 > 0$. Thus there is an integer N such that

$$\left| \frac{a_{n+1}}{a_n} - L \right| < \epsilon = \frac{1 - L}{2}, \quad \text{for } n \geq N,$$

Thus, for all $n \geq N$, $\frac{a_{n+1}}{a_n} - L < \frac{1-L}{2}$, which shows that

$$(3) \quad \frac{a_{n+1}}{a_n} < \frac{1+L}{2} < 1, \quad \text{for } n \geq N.$$

Note that a_n is positive for each n . Hence, if $n \geq N$, then

$$a_{n+1} < a_n.$$

Consequently, $\{a_n\}$ is strictly decreasing for sufficiently large n .

(7 Points) i. *First Proof (3 Points)*

(Using the result from Part (a))

Part (a) shows that $\{a_n\}$ is decreasing. Furthermore, by assumption, we have $a_n > 0$ for all n , implying that the sequence is also bounded below. By the completeness property, we conclude that $\{a_n\}$ has a limit A , say. We now show that $A = 0$ must hold, i.e., we have $\lim a_n = 0$. We argue by a contradiction argument as follows. Suppose that $A \neq 0$. Then $\lim a_n = \lim a_{n+1} = A \neq 0$, and we can apply the quotient rule of the algebraic limit theorems (Theorem 5.1) to find

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{\lim_{n \rightarrow \infty} a_{n+1}}{\lim_{n \rightarrow \infty} a_n} = \frac{A}{A} = 1.$$

However, we also have $\lim(a_{n+1}/a_n) = L < 1$, which contradicts the equality above. Therefore the case $A \neq 0$ cannot occur, and we conclude that $A = 0$ must hold.

ii. *Second Proof (4 Points)*

(Not using the result from Part (a))

Let N be the same as in Part (a). Using Equation (3), we get

$$a_{n+1} < \left(\frac{1+L}{2}\right) a_n,$$

for all $n \geq N$. By induction on n , we have $a_n \leq \left(\frac{1+L}{2}\right)^{n-N} a_N$ for all $n \geq N$. Since a_n is positive for each n and $0 < \frac{1+L}{2} < 1$, then a_n can be arbitrary small when n is large. Therefore,

$$\lim_{n \rightarrow \infty} a_n = 0,$$

as desired.