18.100A – PROBLEM SET #1
SOLUTIONS

1.

(i) Consider $a_n = \frac{\sqrt{n^2-1}}{n} = \sqrt{1-\frac{1}{n^2}}$.

- *Increasing:* We have $a_{n+1} \geq a_n$ for all $n \geq 1$, since

$$\sqrt{1-\frac{1}{(n+1)^2}} \geq \sqrt{1-\frac{1}{n^2}} \quad \Leftrightarrow \quad 1-\frac{1}{(n+1)^2} \geq 1-\frac{1}{n^2} \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \frac{1}{n^2} \geq \frac{1}{(n+1)^2} \quad \Leftrightarrow \quad (n+1)^2 \geq n^2 \quad \Leftrightarrow \quad 2n+1 \geq 0.$$

- *Upper bound:* For example, $b = 1$ is an upper bound, since, for all $n \geq 1$,

$$a_n \leq 1 \quad \Leftrightarrow \quad \sqrt{1-\frac{1}{n^2}} \leq 1 \quad \Leftrightarrow \quad 1-\frac{1}{n^2} \leq 1 \quad \Leftrightarrow \quad 0 \leq \frac{1}{n^2}.$$

- *Limit:* We claim that $a_n \to 1$. Indeed, using $\sqrt{A}-\sqrt{B} = \frac{A^2-B^2}{\sqrt{A}+\sqrt{B}}$, we obtain

$$|a_n - 1| = \Big|\sqrt{1-\frac{1}{n^2}} - 1\Big| = \Big|\frac{1-\frac{1}{n^2}-1}{\sqrt{1-1/n^2}+1}\Big|$$
$$= \frac{1}{n^2}\frac{1}{\sqrt{1-1/n^2}+1} < \frac{1}{n^2}.$$

Hence, for given $\epsilon > 0$,

$$|a_n - 1| < \epsilon, \quad \text{for all } n \geq \tfrac{1}{\sqrt{\epsilon}},$$

proving that $a_n \to 1$.

(ii) Consider $a_n = (2-\frac{1}{n})(2+\frac{1}{n}) = 4 - \frac{1}{n^2}$.

- *Increasing:* We have $a_{n+1} \geq a_n$ for all n, since

$$4-\frac{1}{(n+1)^2} \geq 4-\frac{1}{n^2} \quad \Leftrightarrow \quad -\frac{1}{(n+1)^2} \geq -\frac{1}{n^2} \quad \Leftrightarrow \quad (n+1)^2 \geq n^2.$$

- *Upper bound:* For example, we can take $b = 4$ as an upper bound, since

$$4-\frac{1}{n^2} \leq 4 \quad \Leftrightarrow \quad -\frac{1}{n^2} \leq 0 \quad \Leftrightarrow \quad \frac{1}{n^2} \geq 0.$$

- *Limit:* We claim $a_n \to 4$. To see this, note that, given $\epsilon > 0$, we have

$$|a_n - 4| = \frac{1}{n^2} < \epsilon, \quad \text{if } n > \tfrac{1}{\sqrt{\epsilon}}.$$

(iii) Consider $a_n = \sqrt{n^2+n} - n$.

- *Increasing:* To prove $a_{n+1} \geq a_n$ for all n, we argue as follows:

$$\begin{aligned} & \sqrt{(n+1)^2+(n+1)} - (n+1) \geq \sqrt{n^2+n} - n \\ \Leftrightarrow \quad & \sqrt{(n+1)^2+(n+1)} \geq \sqrt{n^2+n} + 1 \\ \Leftrightarrow \quad & (n+1)^2+(n+1) \geq n^2+n+2\sqrt{n^2+n}+1 \\ \Leftrightarrow \quad & 2n+1 \geq 2\sqrt{n^2+n} \\ \Leftrightarrow \quad & 4n^2+4n+1 \geq 4n^2+4n \\ \Leftrightarrow \quad & 1 \geq 0. \end{aligned}$$

- *Upper bound:* For example, we can take $b = 1/2$, since, for all n,

$$\begin{aligned} & a_n \leq \frac{1}{2} \quad \Leftrightarrow \quad \sqrt{n^2+n} \leq \frac{1}{2} + n \\ \Leftrightarrow \quad & n^2+n \leq \frac{1}{4} + n + n^2 \quad \Leftrightarrow \quad 0 \leq \frac{1}{4}. \end{aligned}$$

- *Limit:* We claim that $a_n \to 1/2$. To show this, we use the pedestrian's approach (i. e. without using algebraic limit theorems etc.) as follows. Again, we first recall $\sqrt{A} - \sqrt{B} = \frac{A^2-B^2}{\sqrt{A}+\sqrt{B}}$. This gives us

$$a_n = \sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}.$$

Thus

$$\begin{aligned} |a_n - \frac{1}{2}| &= \Big|\frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2}\Big| = \Big|\frac{2n - \sqrt{n^2+n} - n}{2\sqrt{n^2+n}+2n}\Big| \\ &= \Big|\frac{n - \sqrt{n^2+n}}{2\sqrt{n^2+n}+2n}\Big| < \frac{1}{4n}|n - \sqrt{n^2+n}|, \end{aligned}$$

using that $\sqrt{n^2+n} \geq n$ in order to estimate the denominator. Note that $|a_n| = |n - \sqrt{n^2+n}|$. Moreover, a_n is a bounded sequence (we found that a_n is bounded above by $1/2$, say, and a_n is also trivially bounded below since $a_n \geq 0$). Therefore,

$$|a_n - \frac{1}{2}| < \frac{1}{4n}\frac{1}{2} = \frac{1}{8n}.$$

Hence, given $\epsilon > 0$, we find

$$|a_n - \frac{1}{2}| < \epsilon, \quad \text{if } n \geq \frac{8}{\epsilon}.$$

(iv) We have $a_n = \sum_{k=1}^{n} \sin^2 k\pi = 0$ for all n, since $\sin(k\pi) = 0$ for every integer k. Thus $a_n = 0$ is clearly increasing, bounded, and converges to 0.

2. Let $a, b \in \mathbb{R}$. We prove $|a+b| + |a-b| \geq |a| + |b|$ as follows.
Note that

$$\begin{aligned} |a+b| &= |a| + |b|, \quad \text{if } \operatorname{sgn} a = \operatorname{sgn} b, \\ |a-b| &= |a| + |b|, \quad \text{if } \operatorname{sgn} a \neq \operatorname{sgn} b. \end{aligned}$$

Since we always have that $|a-b| \geq 0$ and $|a+b| \geq 0$, adding these quantities to these equalities yields the following inequalities:

$$\begin{aligned} |a+b| + |a-b| &\geq |a| + |b|, \quad \text{if } \operatorname{sgn} a = \operatorname{sgn} b, \\ |a-b| + |a+b| &\geq |a| + |b|, \quad \text{if } \operatorname{sgn} a \neq \operatorname{sgn} b. \end{aligned}$$

In summary, this proves our claim.

3. Let $a_n = (1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{n})$. Multiplication yields

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \text{positive terms.}$$

Recall that the sequence $b_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is known to tend to ∞ as $n \to \infty$. Since $b_n \leq a_n$ for all n, this shows that $\{a_n\}$ is not bounded above. Finally, we see that $\{a_n\}$ is increasing because

$$\frac{a_{n+1}}{a_n} = 1 + \frac{1}{n+1} \geq 1,$$

and therefore $a_{n+1} \geq a_n$ for all n, using that a_n is positive.

4.

(a) Let $\{a_n\}$ and $\{b_n\}$ be increasing. Then
  (i) $\{c_n\}$ with $c_n = a_n + b_n$ is increasing as well, since

$$c_{n+1} = a_{n+1} + b_{n+1} \geq a_n + b_n = c_n, \quad \text{for all } n,$$

  using the inequality law that $x+y \geq w+z$ whenever $x \geq w$ and $y \geq z$.
  (i) $\{c_n\}$ with $c_n = a_n - b_n$ is not increasing in general. Counterexample $a_n = n$ and $b_n = 2n$. Then $c_n = -n$, which is clearly not increasing.

(b) Let $\{a_n\}$ and $\{b_n\}$ be bounded above. Obviously, this does not guarantee that $\{a_nb_n\}$ is bounded above; e. g., take $a_n = b_n = -n$ which are bounded above by 0, say. But $a_nb_n = n^2$ is not bounded above anymore.

An additional condition guaranteeing that $\{a_nb_n\}$ is also bounded above would be, for instance, to assume that $a_n \geq 0$ and $b_n \geq 0$. To see this, let A and B be upper bounds for $\{a_n\}$ and $\{b_n\}$, respectively. Note that $A \geq 0$ and $B \geq 0$, since a_n and b_n are non-negative. We can now multiply the inequalities

$$a_n \leq A \quad \text{and} \quad b_n \leq B, \quad \text{for all } n,$$

to conclude that $a_nb_n \leq AB$ for all n. Hence $\{a_nb_n\}$ is bounded above, and $C = AB$ is an upper bound.

(c) Suppose $\{a_n\}$ and $\{b_n\}$ are increasing. Then, in general, the sequence $\{a_nb_n\}$ is not increasing. Counterexample: Take $a_n = n$ and $b_n = -1$ (are constant sequence is always increasing). Then $c_n = a_nb_n = -n$ is not increasing.

An additional condition guaranteeing that $\{a_nb_n\}$ is also increasing would be to require that $a_n \geq 0$ and $b_n \geq 0$, as can be easily seen.

5.

(a) Let $P(n)$ denote the statement that $1 + 3 + \cdots + (2n-1) = n^2$.
  - *Induction Basis:* $P(1)$ is true since $1 = 1^2$.
  - *Induction Step:* Assume that $P(n)$ is true. Then, using that $P(n)$ is true, we conclude

$$1 + 3 + \cdots + (2n-1) + (2(n+1)-1) = n^2 + 2n + 1 = (n+1)^2,$$

    which shows that $P(n+1)$ is also true.
  - *Conclusion:* By induction principle, $P(n)$ is true for all $n \in \mathbb{N}$.

(b) Let $a_{n+1} = \sqrt{1 + \frac{a_n^2}{4}}$, with $0 \le a_1 < 2/\sqrt{3}$.

First, we show that $\{a_n\}$ is bounded above with $a_n < 2/\sqrt{3}$ for all $n \in \mathbb{N}$. We show this by induction as follows.

- *Induction Basis:* $a_1 < 2/\sqrt{3}$ is true by choice of a_1.
- *Induction Step:* Assume that $a_n < 2/\sqrt{3}$ holds for some n. Then
$$a_{n+1} = \sqrt{1 + \frac{a_n^2}{4}} < \frac{2}{\sqrt{3}} \quad \Leftrightarrow \quad 1 + \frac{a_n^4}{4} < \frac{4}{3}$$
$$\Leftrightarrow a_n^2 < \frac{4}{3} \quad \Leftrightarrow \quad a_n < \frac{2}{\sqrt{3}},$$
which shows shat $a_{n+1} < 2/\sqrt{3}$ is also true.
- *Conclusion:* By induction principle, $a_n < 2/\sqrt{3}$ for all $n \in \mathbb{N}$.

Next, we show that $\{a_n\}$ is strictly increasing. Since $a_n \ge 0$ for all n, the statement $a_{n+1} > a_n$ for all n is equivalent to
$$a_{n+1}^2 > a_n^2, \quad \text{for all } n.$$
However, this follows from
$$a_{n+1}^2 - a_n^2 = 1 + \frac{a_n^2}{4} - a_n^2 = 1 - \frac{3}{4}a_n^2 > 0, \quad \text{for all } n,$$
using that $a_n < 2/\sqrt{3}$ for all n, which we have proven previously.

(c) Let $P(n)$ denote the statement that $n^3 - n$ is divisible by 3. We prove this by induction as follows.

- *Induction Basis:* $P(1)$ is true, since 0 is divisible by 3.
- *Induction Step:* Assume that $P(n)$ is true for some n. Then
$$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 2n \\ &= n^3 - n + n + 3n^2 + 2n = n^3 - n + 3(n^2 + n).\end{aligned}$$
Since $n^3 - n$ is divisible by 3 and so is $3(n^2 + n)$, we conclude that $(n+1)^3 - (n+1)$ (which is then a sum of two numbers divisible by 3) is also divisible by 3. Hence $P(n+1)$ is true.
- *Conclusion:* By induction principle, $P(n)$ is true for all $n \in \mathbb{N}$.

6. The induction step contains a flaw. For instance, take $n = 1$ and consider a pack of $n + 1 = 2$ horses. Removing the first or last horse, provides us each with a pack of horses (containing only horse in this case) that have no intersection at all. In this case, the arguments does not imply that both packs share the same color.

Thus $P(1)$ is true does not imply that $P(2)$ is true.