18.100A – PROBLEM SET #7
DUE WEDNESDAY, APR 21, 2008, BY 12:00 NOON

To be handed in class or via the envelope next to Room 2–230.

Note: The 2nd midterm is moved to Monday, Apr 28, in class.

1.

(a) Let
$$f(x) = \begin{cases} 0, & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 1, & \text{otherwise.} \end{cases}$$
Prove that $f(x)$ is integrable on $[0, 1]$.

(b) Let
$$f(x) = \begin{cases} 0, & \text{if } x = m/2^n \text{ for some } m, n \in \mathbb{N}, \\ 1, & \text{otherwise.} \end{cases}$$
Prove that $f(x)$ is not (Riemann-)integrable on $[0, 1]$.

2.

(a) Suppose a function $f(x)$ is integrable on $[a, b]$ and $f(x) = 0$ whenever x is rational. (You are not told anything about the values of $f(x)$ when x is irrational.) Prove that $\int_a^b f(x)\, dx = 0$.

(b) Show that
$$\int_0^{\pi/2} \frac{\sin x}{x(5 + x)}\, dx < \frac{\pi}{10}.$$

3.

(a) Prove that if $f(x)$ is continuous on $I = [a, b]$, $f(x) \geq 0$ for all $x \in I$, and $f(c) > 0$ for some $c \in [a, b]$, then $\int_a^b f(x)\, dx > 0$. (Hint: Use Theorem 19.4B carefully.)

(b) Suppose that for all $a > 0$, $\int_{-a}^a f(t)\, dt = 0$. Prove that $g(x) = \int_0^x f(t)\, dt$ is an even function.

4.

(a) Find the limit
$$\lim_{n\to\infty} \frac{1}{n^{3/2}} \left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}\right).$$
(Hint: Write this as a Riemann sum $\sum_1^n f(x_i')\Delta x_i$ for suitable $f(x)$ defined on a suitable interval $[a, b]$. Hence $\lim_{n\to\infty} \sum_1^n \ldots = \int_a^b f(x)\, dx$ and calculate the integral.)

(b) Prove that
$$\lim_{n\to\infty} \int_1^2 \frac{1}{1 + x^{2n}}\, dx = 0.$$

Note that you cannot just interchange $\lim_{n\to\infty}$ with taking the integral to conclude this! Instead, try using the following argument and work out the details. Let $\varepsilon > 0$ be given. Show that, by choosing $\delta > 0$ suitably, we have

$$\int_1^2 \frac{dx}{1+x^{2n}} = \int_1^{1+\delta} \frac{dx}{1+x^{2n}} + \int_{1+\delta}^2 \frac{dx}{1+x^{2n}} < \varepsilon, \quad \text{for } n \gg 1,$$

whence the claim follows, since we also have that $\int_1^2 \frac{dx}{1+x^{2n}} \geq 0$ holds.

5. Suppose $f(x)$ is a bounded function on $[a,b]$, and $f(x)^2$ is integrable on $[a,b]$. Does it follow that $f(x)$ is integrable? Does the answer change if we assume that $f(x)^3$ is integrable on $[a,b]$?

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