18.100A – PROBLEM SET #5
DUE WEDNESDAY, APR 9, 2008, BY 12:00 NOON

To be handed in class or via the envelope next to Room 2–230.

1. It is well-known that e^x is a continuous function on $\mathbb{R}$. A way to show to this would be as follows. Suppose you know that e^x is continuous at 0. Using the exponential law, show that, for any $x \in \mathbb{R}$, we have $e^{x_n} \to e^x$ whenever $x_n \to x$.

2. Let $f(x)$ be a continuous function on $[0, \infty)$. Show that if

$$f(x) = f(x^2), \quad \text{for all } x \geq 0,$$

then $f(x)$ is a constant function.

(Hint: Use the sequential continuity theorem (Theorem 11.5) and show that $f(0) = f(x)$ for all $0 \leq x < 1$, and $f(1) = f(x)$ for all $x \geq 1$. Finally conclude that $f(0) = f(1)$, which completes the proof.)

3. Suppose that $f(x)$ is continuous on $I = [0, 1]$ and that $f(x) \in I$ for all $x \in I$. (A fancier notation for the latter fact is $f(I) \subset I$.) Prove that $f(x)$ has a fixed point, i. e., there exists $x \in I$ such that $f(x) = x$.

(Hint: Consider $g(x) = f(x) - x$ and use Bolzano's theorem. Make sure that you use somewhere the assumption that $f(I) \subset I$ holds.)

4. Prove that if $f(x)$ is continuous on $I = [0, 1]$ and $f(0) = f(1)$, then for each $n \in \mathbb{N}$ there exists $x \in I$ such that $f(x + 1/n) = f(x)$.

(Hint: Bolzano's theorem...)

5.

(a) Show that if S is the union of a finite number of compact intervals, then S is sequentially compact. (The intervals need not overlap.)
(b) Show by counterexample that this is no longer true if S is the union of an infinite number of compact intervals.
(c) Let $f(x)$ be continuous on $I = [a, b]$ and suppose that $f(x) > 0$ for all $x \in I$. Show that $f(x)$ is in fact uniformly bounded away from zero, i. e., there exists $\delta > 0$ such that $f(x) \geq \delta > 0$ for all $x \in I$. (Hint: Use the Maximum theorem 13.3)
(d) Given an example such that $f(x)$ is continuous on $I = (a, b)$ and $f(x) > 0$ for all $x \in I$, but there is no such $\delta > 0$ as in (c).

6. (Extra Problem.) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and take on every real value (i. e., f is surjective). Suppose that $f(x)$ attains each value *exactly twice*. Prove that such a function $f(x)$ does not exist.

(Hint: Clever use of Bolzano's and Maximum theorem... The assumption that $f(x)$ is surjective is just for convenience; it can be dropped in fact.)

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