18.100A – PROBLEM SET #4
DUE WEDNESDAY, MAR 19, 2008, BY 12:00 NOON

To be handed in class or via the envelope next to Room 2–230.

1. Find the radius of convergence R for the following power series.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{x^n}{2^n \sqrt{n}} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n}}$$

$$\text{(d)} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2} \qquad \text{(e)} \sum_{n=1}^{\infty} \left(\frac{n+2}{n}\right)^n x^n \qquad \text{(f)} \sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}$$

Hint: To determine R, use the ratio test or the n-th root test. Also, you can use the fact (without proof) that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ holds.

2. Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x \in \mathbb{R}; \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

Find the power series for $\frac{e^x}{1-x}$ if $|x| < 1$. That is, use the Multiplication theorem 8.4A for power series to determine c_n such that

$$\sum_{n=0}^{\infty} c_n x^n = \frac{e^x}{1-x}, \quad \text{for } |x| < 1.$$

You do not have to derive a closed formula for c_n; it is sufficient to write down c_0, c_1, $c_2, \ldots$ indicating how c_n is formed.

3.

(a) If f and g are defined for all $x \in \mathbb{R}$ and are odd or even (four possibilities altogether), what can be said about the composition $f \circ g$? Prove it.
(b) Suppose f and g are defined for all $x \in \mathbb{R}$ and are decreasing functions. What if anything can you conclude about the composition $f \circ g$? Prove your claim.
(c) Suppose f is defined for all $x \in \mathbb{R}$. Show that if f is strictly increasing, then f is injective.

4. Let $f(x) = x^2 + 2x + 2$ for $x \in \mathbb{R}$. Determine, if possible, the inverse function f^{-1} for each of the following restrictions of f.

(a) f restricted on $[0, \infty)$.
(b) f restricted on $[-2, 0]$.
(c) f restricted on $(\infty, -2]$.

5.

(a) Find the natural domain for

$$f(x) = \sqrt{9 - \sqrt{25 - \sqrt{x}}}$$

(By natural domain we mean the set of all $x \in \mathbb{R}$ such that the expression defining f above makes sense.)

(b) Can you say anything about the monotonicity of f given in (a)?

6.

(a) Describe all functions $f(x)$ which are defined for all $x \in \mathbb{R}$ and which satisfy $f(x) = 1/f(x)$ for all $x \in \mathbb{R}$. (There are more than two such $f(x)$.)

(b) Prove: Any periodic increasing function defined for all $x \in \mathbb{R}$ is constant. (What about $\tan x$?)

(c) Give an example of a function which is periodic, non-constant, and yet has no minimal period. (Sketch the graph of your example.)

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