18.100A – PROBLEM SET #3
DUE WEDNESDAY, MAR 5, 2008, BY 12:00 NOON

To be handed in class or via the envelope next to Room 2–230.

Note: The first midterm will be on Monday, Mar 10, in Class.

1. Show that $a_n = \sqrt{n}$ defines a sequence whose successive terms get arbitrarily close without being a Cauchy sequence. That is, prove that for any $\epsilon > 0$, there is N such that $|a_n - a_{n+1}| < \epsilon$ for all $n \geq N$.

2.

(a) Suppose $\{x_n\}$ is a sequence that takes only finitely many values $a_1, \ldots, a_k$. That is, for every n, we have $x_n = a_i$ for some i (the index i will depend on n). Prove that $\{x_n\}$ has a cluster point.

(b) Which of these sequences $\{b_n\}$ always have a convergent subsequence, regardless of what $\{a_n\}$ is? Indicate reasoning.

$$\text{i)} \quad b_n = \cos^2 a_n, \qquad \text{ii)} \quad b_n = \frac{a_n}{1+a_n}, \qquad \text{iii)} \quad b_n = \frac{1}{1+|a_n|}.$$

(In ii) we assume that $a_n \neq -1$.)

3. Let $S \subset \mathbb{R}$ be a subset of $\mathbb{R}$. In class, we defined $\sup S$ (the supremum of S) as the least upper bound for S. We also mentioned the completeness property for sets in $\mathbb{R}$: If S is bounded above and non-empty, then $\sup S$ exists.

Likewise, we can define $\inf S$ (the infimum of S) as the greatest lower bound for S. The completeness property can be then stated as follows: If S is bounded below and non-empty, then $\inf S$ exists. (Please see Section 6.5 for more details.)

Let S and T be non-empty and bounded subsets of $\mathbb{R}$, and suppose that for all $s \in S$ and $t \in T$, we have $s \leq t$. Prove that $\sup S \leq \inf T$.

4. The following theorem can be viewed as an "infinite triangle inequality": If $\sum a_n$ is absolutely convergent, then $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$.

(a) Criticize the following "proof": For any integer $k \geq 1$, we have

$$|\sum_{n=1}^{k} a_n| \leq \sum_{n=1}^{k} |a_n|,$$

by the extended triangle inequality (see Section 2.4). By taking the limit $k \to \infty$, we get the infinite triangle inequality.

(b) Give a slightly different proof, by imitating the proof of the usual triangle inequality (see Section 2.4).

5. Test each of the following series for convergence; it may help you to know that $\lim_{n\to\infty}(1+r/n)^n = e^r$.

(a) $\sum_1 \frac{\sqrt{n}}{n^2+1}$ (b) $\sum_1 \frac{n^2}{2^n}$ (c) $\sum_1 \frac{\cos n}{n^2}$ (d) $\sum_0 \frac{(n!)^2}{(2n)!}$

(e) $\sum_1 \left(\frac{n+1}{2n+1}\right)^n$ (f) $\sum_2 \frac{1}{n \ln n}$ (g) $\sum_1 \sin(1/n)$ (h) $\sum_1 \frac{2^n n!}{n^n}$

Here $\sum_N a_n$ stands for $\sum_{n=N}^{\infty} a_n$. You can also use the integral test (see Section 7.5) to discuss convergence.

6*. (Extra Problem) Consider the following theorem: If $\{a_n\}$ is non-negative, decreasing, and $\sum a_n$ converges, then $na_n \to 0$.

(a) Prove this theorem as follows. Show first that:

$$\text{Given } \epsilon > 0, \quad na_{2n} \leq a_{n+1} + \ldots + a_{2n} < \epsilon, \quad \text{for } n \gg 1.$$

Then handle the odd terms a_{2n+1} somehow, and complete the proof.

(b) A converse of the above theorem is: If $na_n \to 0$, $a_n \geq 0$, and a_n is decreasing, then $\sum a_n$ converges. Show that this converse is *false* by giving a counterexample.

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