18.100A – PROBLEM SET #2
DUE WEDNESDAY, FEB 27, 2008, BY 12:00 NOON

To be handed in class or via the envelope next to Room 2–230.

1. Define a sequence $\{a_n\}$ recursively by $a_{n+1} = 2a_n^2$, where $a_0 > 0$.

(a) Show that if $L = \lim a_n$ exists, then either $L = 0$ or $L = 1/2$.

(b) Show that the limit (if it exists) is in general not $1/2$ (i. e., except when $a_0 = 1/2$) by proving that:

$$\lim_{n\to\infty} a_n = \begin{cases} 0 & \text{if } a_0 < 1/2, \\ \infty & \text{if } a_0 > 1/2. \end{cases}$$

(By $\lim a_n = \infty$ we mean that $\{a_n\}$ tends to ∞.)

2. Prove that

$$\lim_{n\to\infty} \sqrt[n]{a} = 1, \qquad \text{if } a > 0,$$

by using the following strategy.

(a) Consider first the case $a > 1$, fix a value of n, put $a^{1/n} = 1 + h_n$, and show that $h_n \to 0$ as $n \to \infty$, by reasoning like that in the proof of Theorem 3.4.

(b) If $a < 1$, then $1/a > 1$; use this and the definition of the limit to deduce this case form the previous one. (Use only the definition and no other "obvious" facts about limits.)

3. Let $a_n = r^n/n!$, and prove that $a_n \to 0$ for all $r \in \mathbb{R}$.

(If $|r| \leq 1$, this is easy. If $|r| > 1$, this is more subtle. Compare two successive terms of the sequence, and show that $|a_{n+1}| < \frac{1}{2}|a_n|$ for $n \gg 1$. Then complete the argument.)

4. Pick a positive number a_0 between 0 and $\pi/2$, take its cosine, then the cosine of that number, and keep on taking cosines. You get a sequence $\{a_n\}$ given by $a_{n+1} = \cos a_n$.

(a) Try it on a calculator a few times. What eight place decimal number do you end up with? Assume that $\{a_n\}$ has a limit L. What equation does L solve?

(b) Assume that $a_n \to L$ and define the error term $e_n = a_n - L$. Prove that

$$|e_{n+1}| < K|e_n|, \quad \text{for } n \gg 1,$$

where $0 < K < 1$ is some constant. (Hint: The identity $\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$ and the inequality $|\sin x| \leq |x|$ might come in handy.)

5. Suppose $a_n/b_n \to L$, with $b_n \neq 0$ for all n, and $b_n \to 0$. Prove that this implies $a_n \to 0$. (Here, as always, L represents a finite number, not ∞.)

6. Explain why we can assume that $\epsilon > 0$ is sufficiently small when proving the algebraic limit theorems (see Chapter 5.1). For instance, in order to show the product rule,

$$a_n \to M \quad \text{and} \quad b_n \to L \quad \Rightarrow \quad a_n b_n \to ML,$$

we assume in its proof that $\epsilon < 1$ holds. Why is this legitimate?

7. Assume $a_{n+1}/a_n \to L$, where $L < 1$ and $a_n > 0$. Prove that:

(a) $\{a_n\}$ is decreasing for $n \gg 1$;
(b) $a_n \to 0$. (Give two proofs; one using (a), and a direct one.)

Enno Lenzmann, M.I.T., Department of Mathematics, Room 2–230
E-mail address: lenzmann@math.mit.edu