18.100A – PROBLEM SET #1
DUE THURSDAY, FEB 19, 2008, BY 12:00 NOON

To be handed in class or via the envelope next to Room 2–230.
Note that Monday, Feb 18, is President's Day. Hence this pset is due Tuesday, Feb 19.

1. Consider the following sequences $\{a_n\}$. Show that each $\{a_n\}$ is increasing; find an upper bound, if it exists; give the limit if you can.

(i) $a_n = \dfrac{\sqrt{n^2-1}}{n}$,

(ii) $a_n = \left(2 - \dfrac{1}{n}\right)\left(2 + \dfrac{1}{n}\right)$,

(iii) $a_n = \sqrt{n^2+n} - n$,

(iv) $a_n = \displaystyle\sum_{k=1}^{n} \sin^2 k\pi$.

2. Prove that $|a+b| + |a-b| \geq |a| + |b|$ for all $a, b \in \mathbb{R}$.

3. Prove that $a_n = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)$ is strictly increasing, but not bounded above. (Hint: You may find Proposition 1.5A on page 8 in the textbook useful.)

4.

(a) Let $\{a_n\}$ and $\{b_n\}$ be increasing; are the following increasing? Proof or counterexample.

$$\text{(i)} \quad \{a_n + b_n\}, \qquad \text{(ii)} \quad \{a_n - b_n\}.$$

(b) Let $\{a_n\}$ and $\{b_n\}$ be bounded above. State hypotheses which guarantee that $\{a_n b_n\}$ will also be bounded above, and prove it.

(c) "If $\{a_n\}$ and $\{b_n\}$ are increasing, then $\{a_n b_n\}$ is increasing". Show that this is false, make the hypotheses stronger, and prove the amended statement.

5. Read Section A.4 on mathematical induction in Mattuck's textbook "Introduction to Analysis". Apply the principle of induction to solve the following problems.

(a) Prove that $1 + 3 + 5 + \ldots + (2n-1) = n^2$ for all $n \in \mathbb{N}$.

(b) Let $\{a_n\}$ be defined by

$$a_{n+1} = \sqrt{1 + a_n^2/4}, \quad \text{with } 0 \leq a_1 < 2/\sqrt{3}.$$

Show that $\{a_n\}$ is strictly increasing and bounded above. (Hint: Prove first that $\{a_n\}$ is bounded above by using induction.)

(c) Prove the following instance of *Fermat's Little Theorem:* For any $n \in \mathbb{N}$, the number $n^3 - n$ is an integer multiple of 3. (Remark: In general Fermat's Little Theorem states that, for any $n \in \mathbb{N}$, the number $n^p - n$ is an integer multiple of p, provided that p is prime.)

6. Find the logical flaw in the following argument which proves:

"All horses are the same color."

"Proof." Let $P(n)$ denote the proposition that n horses are the same color. Clearly $P(1)$ is true and hence the basis step is complete.

To prove the induction step, i. e., $P(n)$ implies that $P(n+1)$, we argue as follows. Assume that $P(n)$ is true. Given any set of $n+1$ horses, we number the horses from 1 to $n+1$, and we remove the last horse labeled with $n+1$. Since $P(n)$ is true, we see that the first n horses are the same color. But by excluding the first horse in the pack of $n+1$ horses, we can conclude that the last n horses also share the same color. Therefore all $n+1$ horses are the same color. QED.

Enno Lenzmann, M.I.T., Department of Mathematics, Room 2–230
E-mail address: lenzmann@math.mit.edu