18.103 PROBLEM SET #7
DUE FRIDAY, NOV 21, 2008, 11:00AM

To be turned in during class or via the envelope next to Room 2–378.

1. Suppose that $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a proper rotation.

(a) Show that $p(t) = \det(R - tI)$ (where I is the 3×3-identity matrix) is a polynomial of degree 3, and prove that there exists $\gamma \in S^2$ (where S^2 denotes the unit sphere in $\mathbb{R}^3$) with

$$R(\gamma) = \gamma.$$

[Hint: Use the fact that $p(0) > 0$ to see that there is $\lambda > 0$ with $p(\lambda) = 0$. Then $R - \lambda I$ is singular, so its kernel is non-trivial.]

(b) If $\mathcal{P}$ denotes the plane perpendicular to γ and passing through the origin, show that

$$R : \mathcal{P} \to \mathcal{P},$$

and that this linear map is a rotation.

2.

(a) Let A be a $d \times d$ positive definite matrix with real coefficients. Show that

$$\int_{\mathbb{R}^d} e^{-\pi x \cdot Ax}\, dx = \frac{1}{\sqrt{\det A}}.$$

This generalizes the fact that $\int_{\mathbb{R}^d} e^{-\pi|x|^2}\, dx = 1$, which clearly corresponds to the case where A is the identity. [Hint: Apply the spectral theorem to write $A = RDR^{-1}$ where R is a rotation, D is diagonal with entries $\lambda_1, \ldots, \lambda_d$, corresponding to the eigenvalues of A.]

(b) Prove the identity

$$e^{-\beta} = \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-\beta^2/4u}\, du$$

when $\beta \geq 0$. [Hint: Take the Fourier transform of both sides with $\beta = 2\pi|x|$.]

3. For every real number $a > 0$, we define the **fractional Laplacian** $(-\Delta)^a$ by the formula

$$(-\Delta)^a f(x) = \int_{\mathbb{R}^d} (2\pi|\xi|)^{2a} \hat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi$$

whenever $f \in \mathcal{S}(\mathbb{R}^d)$. It is easy to see that $(-\Delta)^a$ agrees with the usual definition of the a-th power of $-\Delta$ (i.e., a compositions os minus the Laplacian) when a is a positive integer.

(a) Verify that $(-\Delta)^a(f)$ is a smooth function.

(b) Prove that if a is not an integer, then in general $(-\Delta)^a(f)$ is not rapidly decreasing. Thus $(-\Delta)^a : \mathcal{S} \not\to \mathcal{S}$ whenever a is not an integer.

4. Recall that the X**-ray transform**, or also called **Radon transform** in $\mathbb{R}^2$, of a function $f \in \mathcal{S}(\mathbb{R}^2)$ is defined as

$$X(f)(L) = \int_L f,$$

where L is a (straight) line in $\mathbb{R}^2$, and the integral is taken along L (which will be detailed below). Note that this transform assigns to each $fm \in \mathcal{S}(\mathbb{R}^2)$ another function $X(f)$ whose domain is the set of lines in $\mathbb{R}^2$.

(a) We now specify how $\int_L f$ can be actually defined. For each (t, θ) with $t \in \mathbb{R}$ and $|\theta| \leq \pi$, let $L = L_{t,\theta}$ denote the line in $\mathbb{R}^2$ given by

$$x \cos\theta + y \sin\theta = t.$$

This is the line perpendicular to the direction $(\cos\theta, \sin\theta)$ at "distance" t from the origin (we allow negative t also). For $f \in \mathcal{S}(\mathbb{R}^2)$ the two-dimensional Radon transform $X(f)$ is given by

$$X(f)(t,\theta) = \int_{L_{t,\theta}} f = \int_{-\infty}^{\infty} f(t\cos\theta + u\sin\theta, t\sin\theta - u\cos\theta)\, du.$$

Calculate $X(f)$ for $f(x,y) = e^{-\pi(x^2+y^2)}$.

(b) Show that if $X(f) = 0$ for $f \in \mathcal{S}(\mathbb{R}^2)$, then $f \equiv 0$, by taking the Fourier transform in one variable.

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