18.103 PROBLEM SET #6
DUE FRIDAY, NOV 14, 2008, 11:00AM

To be turned in during class or via the envelope next to Room 2–378.

1. Prove the following uniqueness theorem for harmonic functions in the strip $\{(x, y) : 0 < y < 1, -\infty < x < \infty\}$: if u is harmonic in the strip, continuous on its closure with $u(x, 0) = u(x, 1) = 0$ for all $x \in \mathbb{R}$, and u vanishes at infinity, then $u \equiv 0$.

Moreover, show that the condition "u vanishes at infinity" cannot be dropped, by giving an example such that $u \not\equiv 0$. [Hint: Consider $u(x, y) = f(x)g(y)$ and find appropriate f and g so that $u(x, 0) = u(x, 1) = 0$ and $\Delta u = 0$.]

2. The **zeta function** is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$. Verify the identity

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty t^{\frac{s}{2}-1}(\vartheta(t) - 1)\, dt \quad \text{for } s > 1,$$

where Γ and ϑ are the gamma and theta functions, respectively:

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\, dt \quad \text{and} \quad \vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}.$$

3. Suppose $f \in \mathcal{S}(\mathbb{R})$. Show that f and $\hat{f}$ cannot both be compactly supported unless $f \equiv 0$.

[Hint: Assume $f \not\equiv 0$ is supported in $[0, 1/2]$. Expand f is an Fourier Series in the interval $[0, 1]$, and note that as a result, f is a trigonometric polynomial.]

4. Let g be defined by

$$g(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Since $g \in L^1(\mathbb{R})$, it makes sense to consider its Fourier transform given by $\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i x\xi}\, dx$. Show that

$$\hat{g}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2.$$

(b) Assume that the Poisson summation formula is still valid for g, prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n + \alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}$$

whenever $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

(c) Prove as a consequence of (b) that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n + \alpha} = \frac{\pi}{\tan \pi\alpha}$$

whenever $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. [Hint: First prove this when $0 < \alpha < 1$. To do so, integrate the formula in (c). What is the precise meaning of the series on the left side? Evaluate at $\alpha = 1/2$.]

5. The Heisenberg uncertainty principle can be formulated in terms of the operator $L = -\frac{d^2}{dx^2} + x^2$, which acts on $\mathcal{S}(\mathbb{R})$ by the formula

$$L(f) = -\frac{d^2 f}{dx^2} + x^2 f.$$

Sometimes L is called the **Hermite operator** or the **Hamilton operator of the quantum harmonic oscillator** (in one space dimension). Consider the usual inner product on $\mathcal{S}(\mathbb{R})$ given by

$$(f, g) = \int_{-\infty}^{\infty} \overline{f}(x) g(x)\, dx.$$

(a) Prove that the Heisenberg uncertainty principle implies

$$(Lf, f) \geq (f, f) \quad \text{for all } f \in \mathcal{S}(\mathbb{R}).$$

This is usually denoted by $L \geq I$ where I is the identity operator. [Hint: Integrate by parts.]

(b) Consider the operators A and A^* defined on $\mathcal{S}$ by

$$A(f) = \frac{df}{dx} + xf \quad \text{and} \quad A^*(f) = -\frac{df}{dx} + xf.$$

The operators A and A^* are sometimes called the **annihilation** and **creation** operators, respectively. Prove that, for all $f, g \in \mathcal{S}(\mathbb{R})$, we have the following.

(i) $(Af, g) = (f, A^* g)$.
(ii) $(Af, Af) = (A^* Af, f) \geq 0$.
(iii) $A^* A = L - I$.

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