18.103 PROBLEM SET #5
DUE MONDAY, OCT 27, 2008, 11:00AM

To be turned in during class or via the envelope next to Room 2–378.

1. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and let $E \in \mathcal{M}$ be a measurable set. Prove the following statements.

(a) *(Chebyshev's inequality).* If $f \geq 0$ is a measurable, then

$$\mu(E_\lambda) \leq \frac{1}{\lambda} \int_E f \, d\mu,$$

where $E_\lambda = \{x \in E : f(x) \geq \lambda\}$ for $\lambda > 0$.

(b) If $f \geq 0$ is measurable and $\int_E f \, d\mu < +\infty$, then $f(x) < +\infty$ for almost every $x \in E$.

(c) If $f \geq 0$ is measurable and $\int_E f \, d\mu = 0$, then $f(x) = 0$ for almost every $x \in E$.

[Hints: For (b), consider $A = \{x \in E : f(x) = +\infty\}$ and apply Chebyshev's inequality to the sets $A_n = \{x \in E : f(x) \geq n\}$ for $n \in \mathbb{N}$, and show that $\mu(A) = 0$. As for (c), apply Chebeshev to $A_n = \{x \in E : f(x) \geq 1/n\}$ and show that $A = \cup_{n=1}^{\infty} A_n$ has measure zero.]

2. Let $f \in \mathcal{S}(\mathbb{R})$ and suppose that $\int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy} \, dy = 0$ for all $x \in \mathbb{R}$. Show that $f \equiv 0$. [Hint: Consider $f * e^{-x^2}$.]

3.

(a) Let f be a continuously differentiable on $[0, 1]$. Show that

$$\sup_{0 \leq x, y \leq 1} |f(x) - f(y)| \leq \Big(\int_0^1 |f'(x)|^2 \, dx \Big)^{1/2}$$

[Hint: Use the Cauchy-Schwarz inequality]

(b) For $f \in L^2(\mathbb{R})$ and $M > 0$, define

$$f_M(x) = \begin{cases} f(x) & \text{when } |x| \leq M, \\ 0 & \text{when } |x| > M. \end{cases}$$

Prove that $\|f_M - f\|_{L^2} \to 0$ as $M \to \infty$.

4. The following exercise illustrates the principle that the decay of $\hat{f}$ is related to the continuity properties of f.

(a) Suppose that f is a function of moderate decrease on $\mathbb{R}$ (see Stein 5.1.1) whose Fourier transform $\hat{f}$ is continuous and satisfies

$$\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}} \right) \quad \text{as} \quad |\xi| \to \infty,$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hölder condition of order α, i. e.,

$$|f(x + h) - f(x)| \leq M |h|^\alpha \quad \text{for some } M > 0 \text{ and all } x, h \in \mathbb{R}.$$

(b) Let f be a continuous function on $\mathbb{R}$ which vanishes identically for $|x| \geq 1$, with $f(0) = 0$, and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of the origin. Prove that $\hat{f}$ is *not* of moderate decrease. In fact, there is no $\epsilon > 0$ so that $\hat{f} = O(1/|\xi|^{1+\epsilon})$ as $|\xi| \to \infty$.

[Hint: For part (a), use the Fourier inversion formula to express $f(x+h) - f(x)$ as an integral involving $\hat{f}$, and estimate this integral separately for ξ in the two regions $|\xi| \leq 1/|h|$ and $\xi \geq 1/|h|$.]

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