18.103 – PROBLEM SET #4
DUE WEDNESDAY, OCT 8, 2008, 11:00AM

To be turned in during class or via the envelope next to Room 2–378.

Announcement: Our first midterm exam is scheduled for Friday, Oct 10, 2008, in class.

Remark: This problem set is rather short. So you should have enough time to study for the upcoming midterm.

1. Let $f \in C^1(\mathbb{S}^1)$. Show that the Fourier series of f is absolutely convergent. [Hint: Use the Cauchy-Schwarz inequality and Parseval's identity for f'.]

2. Let μ be the Lebesgue measure on $\mathbb{R}$.

(a) Show that $\mathbb{Z} \in \mathcal{M}_F$ by finding a sequence $\{A_n\}$ of elementary sets such that $d(\mathbb{Z}, A_n) \to 0$ as $n \to \infty$. Also, conclude that $\mu(\mathbb{Z}) = 0$ holds.

(b) Prove that for any $\epsilon > 0$ there exists an open dense subset $A \subset \mathbb{R}$ such that $\mu(A) < \epsilon$. [Hint: Argue that you can find open set $A \supset \mathbb{Q}$ with the desired property.]

(c) Prove the *translational invariance* of μ. That is, for any given subset $A \subset \mathbb{R}$ and real number $c \in \mathbb{R}$, let $A + c = \{w + c : w \in A\}$. Prove that, if $A \in \mathcal{M}$, then $A + c \in \mathcal{M}$ and

$$\mu(A + c) = \mu(A). \tag{$*$}$$

[Hint: By definition of μ for elementary sets, the translational invariance immediately follows for $A \in \mathcal{E}$. Hence, it remains to show that in equation $(*)$ the outer measures on both sides are equal.]

Enno Lenzmann, M.I.T., Department of Mathematics, Room 2–378, Office Hours: Wed, 3–4pm or by appointment.

E-mail address: `lenzmann@math.mit.edu`