

18.103 – PROBLEM SET #3
DUE WEDNESDAY, OCT 1, 2008, 11:00AM

To be handed in class or via the envelope next to Room 2–378.

1. Prove that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

[Hint: Start with the fact that the integral of $D_N(\theta)$ equals 2π , and note that the difference $(1/\sin(\theta/2)) - 2/\theta$ is continuous on $[-\pi, \pi]$. Apply the Riemann-Lebesgue lemma.]

2. Suppose that f is periodic and of class C^k . Show that

$$\hat{f}(n) = o(1/|n|^k).$$

That is, $|n|^k \hat{f}(n) \rightarrow 0$ as $|n| \rightarrow \infty$. [Hint: Using integration by parts, derive a formula for the Fourier coefficients of $f^{(k)}$ and apply the Riemann-Lebesgue lemma to conclude.]

3. Let f be 2π -periodic and Riemann integrable.

(a) Show that

$$\hat{f}(n) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + x/n) e^{-inx} dx$$

and hence

$$\hat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [f(x) - f(x + \pi/n)] e^{-inx} dx.$$

(b) Now assume that f satisfies a Hölder condition of order α , namely

$$|f(x+h) - f(x)| \leq C|h|^\alpha$$

for some $0 < \alpha \leq 1$, some $C > 0$, and all x, h . Use part (a) to show that

$$\hat{f}(n) = O(1/|n|^\alpha).$$

4. Let X be an infinite set and let \mathcal{R} be the following collection of subsets: $A \in \mathcal{R}$ if and only if A is finite or its complement A^c is finite. Let μ be the following set function on \mathcal{R} : $\mu(A) = 0$ if A is finite, and $\mu(A) = 1$ if A^c is infinite. Is μ a measure?

5. Let X be an uncountable set. Let \mathcal{R} be the collection of all finite subsets of X . Given $A \in \mathcal{R}$, let $\mu(A)$ be the number of elements A . Show that \mathcal{R} is a ring and that μ is a measure on \mathcal{R} (i.e., the set function $\mu : \mathcal{R} \rightarrow [0, \infty]$ is countably additive.) Identify the associated outer measure μ^* . What are \mathcal{M} and \mathcal{M}_F ? Is every subset of X measurable?

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