18.103 – PROBLEM SET #2
DUE FRIDAY, SEPT 19, 2008, 11:00AM

To be handed in class or via the envelope next to Room 2–378.

1. Let $P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$ denote the Poisson kernel. Show that the function

$$u(r,\theta) = \frac{\partial P}{\partial \theta},$$

defined for $0 \le r < 1$ and $\theta \in \mathbb{R}$, satisfies:

(i) $\triangle u = 0$ in the disc $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.
(ii) $\lim_{r\to 1} u(r,\theta) = 0$ for each θ.

However, u is not identically zero. Why is the limit in (ii) not uniform? Give reasons.

2. Recall the Riemann-Lebesgue lemma from class:

$$\text{If } f \in \mathcal{R}(S^1), \text{ then } \hat{f}(n) \to 0 \text{ as } |n| \to \infty.$$

We proved this claim for continuous $f \in C^0(S^1)$, by using the Weierstrass approximation theorem (in the form of Corollary 5.4 in Stein & Shakarchi). Complete the proof of the Riemann-Lebesgue lemma for Riemann integrable functions $f \in \mathcal{R}(S^1)$.

[Hint: Use the approximation Lemma 3.2 in Stein & Shakarchi.]

3. Let

$$f(\theta) = \begin{cases} 0 & \text{for } \theta = 0, \\ \log(1/\theta) & \text{for } 0 < \theta \le 2\pi, \end{cases}$$

and define a sequence of functions in $\mathcal{R}$ by

$$f_n(\theta) = \begin{cases} 0 & \text{for } 0 \le \theta \le 1/n, \\ f(\theta) & \text{for } 1/n < \theta \le 2\pi. \end{cases}$$

Prove that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{R}$ with respect to the the L^2-norm. However, f does not belong to $\mathcal{R}$. (You don't have to prove the latter fact.)

[Hint: Show that $\int_a^b (\log\theta)^2\, d\theta \to 0$ if $0 < a < b$ and $b \to 0$, by using the fact that the derivative of $\theta(\log\theta)^2 - 2\theta\log\theta + 2\theta$ is equal to $(\log\theta)^2$.]

4. Consider the sequence $\{a_k\}_{k=-\infty}^{\infty}$ defined by

$$a_k = \begin{cases} 1/k & \text{if } k \ge 1, \\ 0 & \text{if } k \le 0. \end{cases}$$

Note that $\{a_k\} \in \ell^2(\mathbb{Z})$. Prove that there is no $f \in \mathcal{R}$ such that $\hat{f}(k) = a_k$ for all k.

[Hint: Argue by contradiction and assume that such an $f \in \mathcal{R}$ exists and consider the Abel means $A_r(f)(\theta)$ as $r \to 1$.]

5. Show that for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, the Fourier series of

$$\frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}$$

on $[0, 2\pi]$ is given by

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} e^{inx}.$$

Furthermore, apply Parseval's formula to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}.$$

6. Prove the following inequality due to Wirtinger and Poincaré and find the cases of equality:

(a) If f is C^1, has period $T > 0$, and vanishing mean $\int_0^T f(t)\,dt = 0$, then

$$\int_0^T |f(t)|^2\,dt \leq \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2\,dt.$$

(b) We have equality in (a) if and only if $f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T)$, where A, B are some constants.

[Hint: Apply Parseval's identity.]

7 (Extra Problem). Prove the following version of *Fejér's lemma:* For any f and g in $\mathcal{R}(S^1)$, we have

$$\lim_{\Omega \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(\Omega t)\,dt = \hat{f}(0)\hat{g}(0).$$

[Hint: Approximate f by polynomials or step functions.]

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