18.103 – PROBLEM SET #1
DUE FRIDAY, SEPT 12, 2008, 11:00AM

To be handed in class or via the envelope next to Room 2–378.

1. The solutions to the following exercises (a)–(c) can be found in many textbooks on analysis. However, I do recommend that you first make a serious attempt before you copy any examples or proofs.

(a) Give an example of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined $[0, 1]$ such that $f_n \to f$ uniformly on $[0, 1]$.
(b) Prove that if a sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function f on $[a, b]$, then f is continuous on $[a, b]$.
(c) Prove that if f is continuous on the compact interval $[a, b]$, then f is uniformly continuous in $[a, b]$, i. e., for any $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in [a, b]$ such that $|x - y| < \delta$.

2. Prove that if a series $\sum c_n$ converges to s, then $\sum c_n$ is Cesàro summable to s.
[Hint: Assume $s_n \to 0$ as $n \to \infty$. Why is it enough to prove the claim when $s = 0$?]

3. The purpose of this problem is to prove that Abel summability is stronger than the standard or Cesàro methods of summation.

(a) Show that if the series $\sum c_n$ converges to s, then $\sum c_n$ is Abel summable to s. [Hint: Why is it enough (again) to prove this when $s = 0$? Assuming $s = 0$, show that if $s_N = c_1 + \cdots + c_N$, then $\sum_{n=1}^{N} c_n r^n = (1-r) \sum_{n=1}^{N} s_n r^n + s_N r^{N+1}$. Let $N \to \infty$ to deduce that
$$\sum c_n r^n = (1 - r) \sum s_n r^n.$$
Finally, prove that the right side converges to 0 as $r \to 1$.]
(b) However, show that there exist series which are Abel summable, but that do not converge. [Hint: Try $c_n = (-1)^n$. What is the Abel limit of $\sum c_n$?]
(c) Argue similarly to prove that if a series $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to σ, then it is Abel summable to σ. [Hint: Note that
$$\sum_{n=1}^{\infty} c_n r^n = (1 - r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n,$$
and assume $\sigma = 0$.]
(d) Give an example of a series that is Abel summable but not Cesàro summable. [Hint: Try $c_n = (-1)^{n-1} n$. Note that if $\sum c_n$ is Cesàro summable, then c_n/n tends to 0.]

Remark. The results of Problems 2 and 3 can be summarized by the following implications about series:

$$\text{convergent} \quad \Rightarrow \quad \text{Cesàro summable} \quad \Rightarrow \quad \text{Abel summable},$$

and the fact the none of the arrows can be reversed.

4. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of Riemann integrable functions on the interval $[0,1]$ such that

$$\int_0^1 |f_k(x) - f(x)|\, dx \to 0 \quad \text{as} \quad k \to \infty.$$

Show that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$. That is, for any $\epsilon > 0$, there is $k_0 \geq 1$ such that $|\hat{f}_k(n) - \hat{f}(n)| < \epsilon$, for all $n \in \mathbb{N}$ whenever $k \geq k_0$.

5. Recall from Section 5.4 that the Abel means of f converge to f at all point of continuity; that is,

$$\lim_{r \to 1} A_r(f)(\theta) = \lim_{r \to 1} (P_r * f)(\theta) = f(\theta), \quad \text{with } 0 < r < 1,$$

whenever f is continuous at θ. In this problem, we study the behavior of $A_r(f)(\theta)$ at certain points of discontinuity.

A function f is said to have a **jump discontinuity** at θ if the two limits

$$\lim_{\substack{h \to 0 \\ h > 0}} f(\theta + h) = f(\theta^+) \quad \text{and} \quad \lim_{\substack{h \to 0 \\ h > 0}} f(\theta - h) = f(\theta^-)$$

both exist and $f(\theta^+) \neq f(\theta^-)$ holds.

(a) Prove that if f has a jump discontinuity at θ, then

$$\lim_{r \to 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}, \quad \text{with } 0 \leq r < 1.$$

[Hint: Explain why $\frac{1}{2\pi} \int_{-\pi}^{0} P_r(\theta)\, d\theta = \frac{1}{2\pi} \int_0^{\pi} P_r(\theta)\, d\theta = \frac{1}{2}$, then modify the proof of Theorem 4.1 given in the textbook accordingly.]

6. One can construct Riemann integrable functions on $[0,1]$ that have a dense set of discontinuities as follows.

(a) Let $f(x) = 0$ when $x < 0$, and $f(x) = 1$ if $x \geq 0$. Choose a countable dense sequence $\{r_n\}$ in $[0,1]$. (For example, let $\{r_n\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$.) Then, show that the function

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n),$$

is integrable and has discontinuities at all points of the sequence $\{r_n\}$. [Hint: F is monotonic and bounded.]

(b) **(Extra Problem)** Consider next

$$F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n),$$

where $g(x) = \sin(1/x)$ when $x \neq 0$, and $g(0) = 0$. Then F is integrable, discontinuities at each $x = r_n$, and fails to be monotonic in any subinterval of $[0,1]$. [Hint: Use the fact that $3^{-k} > \sum_{n>k} 3^{-n}$.]

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