18.103 HANDOUT #1: FOURIER SERIES FOR $\mathcal{R}(\mathbb{S}^1)$

So far, we have considered Fourier Series for Riemann integrable functions on the unit circle $\mathbb{S}^1 = \{(x,y) : x^2 + y^2 = 1\}$. That is, all the functions f belong to the space

$$\mathcal{R}(\mathbb{S}^1) = \{f : \mathbb{S}^1 \to \mathbb{C} : f \text{ is Riemann integrable}\}.$$

Likewise, we defined $C^k(\mathbb{S}^1)$ to be the set of k times continuously differentiable functions on $\mathbb{S}^1$; in particular, the set $C^0(\mathbb{S}^1)$ denotes the set of continuous f defined on $\mathbb{S}^1$. Also, recall that working with $\mathbb{S}^1$ is just for our convenience, and it is easy to see that $\mathcal{R}(\mathbb{S}^1)$ is trivially identifiable with space of 2π-periodic Riemann integrable functions on $\mathbb{R}$; and, more generally, with the space of Riemann integrable functions with some given period $L > 0$.

1. Very Basic Convergence Results

Fourier Coefficients, Fourier Series, and Convolution. We parametrize $\mathbb{S}^1$ by $x = \cos\theta$ and $y = \sin\theta$ with $\theta \in [-\pi, \pi]$. Hence any $f \in \mathcal{R}(\mathbb{S}^1)$ can be viewed as a function on $[-\pi,\pi]$ such that $f(-\pi) = f(\pi)$. We shall do so in the following.

For each $f \in \mathcal{R}(\mathbb{S}^1)$, we define its *Fourier coefficients* by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta, \quad \text{for } n \in \mathbb{Z}.$$

Accordingly, we associate to f its Fourier series and denote this by

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

Note that, at this stage, it is not clear what is actually meant by the series on the right-hand side. To this end, we introduce the N-th partial (symmetric) Fourier sums

$$S_N(f)(\theta) = \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}.$$

Hence the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}$ will always be interpreted in the sense that we study the limit of the partial sums $S_N(f)$ as $N \to \infty$.

Next, given two functions f and g in $\mathcal{R}(\mathbb{S}^1)$, we define their *convolution* as

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi - \theta)g(\phi)\,d\phi.$$

Note that $f*g$ gives rise to another (Riemann integrable) function on $\mathbb{S}^1$. Moreover, we have the following properties:

- $f * (g + h) = (f * g) + (f * h)$.
- $(cf) * g = c(f * g) = f * (cg)$ for any constant $c \in \mathbb{C}$.
- $f * g = g * f$.
- $(f * g) * h = f * (g * h)$.
- $f * g \in C^0(\mathbb{S}^1)$; i. e., $f * g$ is a continuos function.
- $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.

Except for the last two properties, the proofs are very straightforward. See Stein & Shakarchi, p45, for a complete proof of the last two properties.

Using now the definition of $*$, we can re-write the partial Fourier sums conveniently as follows.

$$S_N(f)(\theta) = (f * D_N)(\theta).$$

Here $D_N(\theta)$ is **Dirichlet kernel** and it is found to be

$$\boxed{D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta} = \frac{\sin((N+1/2)\theta)}{\sin(\theta/2)}}$$

Good Kernels. A family $\{K_n(x)\}_{n=1}^{\infty}$ of functions in $\mathcal{R}(\mathbb{S}^1)$ is said to be family of **good kernel** (or **approximate identities**) if the following three properties are satisfied:

(1) *Normalization:* For all $n \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1.$$

(2) *Boundedness:* There exists a constant $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)| \leq M.$$

(3) *Concentration at Origin:* For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\,dx \to 0 \quad \text{as} \quad n \to \infty.$$

Remark. In practice, we often encounter nonnegative $K_n(x) \geq 0$, in which the boundedness property follows from the normalization property. An easy example for a family of good kernels (which are in fact nonnegative) is given by

$$K_n(x) = \begin{cases} n & \text{for } |x| \leq \pi/n, \\ 0 & \text{for } |x| > \pi/n. \end{cases}$$

Remark. The family $\{D_N\}_{N=1}^{\infty}$, with $D_N(x)$ being the Dirichlet kernel, is **not a family of good kernels**. This fact makes Fourier analysis so intricate!

The next result clarifies why good kernels are sometimes called approximate identities.

Theorem 1. *Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels, and suppose $f \in \mathcal{R}(\mathbb{S}^1)$. Then we have*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. Moreover, the limit above is uniform in x provided that $f \in C^0(\mathbb{S}^1)$ holds. This means that, for $f \in C^0(\mathbb{S}^1)$ and $\epsilon > 0$, there is $N \geq 1$ such that

$$|f(\theta) - (f * K_n)| < \epsilon \quad \textit{for all } \theta \in [-\pi, \pi],$$

whenever $n \geq N$.

Fejér's Theorem. A basic convergence result for Fourier Series is Fejér's theorem, which states that the *Cesàro means* of $S_N(f)(\theta)$ converge for every θ. This will lead to important conclusions.

Let us define the concept of Cesàsor summability first: Given a series $\sum_{k=0}^{\infty} c_k$, let $s_n = c_0 + \cdots + c_n$ denote its sequence of partial sums. We define the sequence $\{\sigma_N\_N = 1^{\infty}$ of Cesáro means by taking the arithmetic means:

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

We say that $\sum_{k=0}^{\infty} c_k$ is **Cesàro summable** to σ, if the limit $\sigma = \lim_{N\to\infty} \sigma_N$ exists. Unsurprisingly, we have that Cesáro summability generalizes the usual summability of series in the sense that

$$\sum_{k=0}^{\infty} c_k \text{ is convergent} \quad \Rightarrow \quad \sum_{k=0}^{\infty} c_k \text{ is Cesásor summable.}$$

Example. The (divergent) series

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots$$

is Cesáro summable with $\sigma_N \to 1/2$.

Let us now consider the Cesáro means for $S_N(f)(\theta)(\theta)$ which are given by

$$\sigma_N(f)(\theta) = \frac{S_0(f)(\theta) + \cdots + S_{N-1}(f)(\theta)}{N}.$$

Since $S_n(f) - f * D_n$, we find that

$$\sigma_N(f)(\theta) = (f * F_N)(\theta),$$

where $F_N(x)$ is **Fejér kernel**. We obtain (using basic trigonometric identities) the formula

$$\boxed{F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}}$$

Furthermore, we verify that $F_N(x) \geq 0$ and, more importantly, that $\{F_N(x)\}_{N=1}^{\infty}$ is a family of good kernels. Thus we see that Theorem 1 immediately yields the following fact.

Theorem 2. (Fejér) *Let $f \in \mathcal{R}(\mathbb{S}^1)$. Then the Fourier series of f is Cesáro summable at every point of continuity of f. That is,*

$$\lim_{N\to\infty} \sigma_N(f)(\theta) = f(\theta)$$

whenever f is continuous at θ. Moreover, the limit above is uniform in θ provided that $f \in C^0(\mathbb{S}^1)$ holds.

As two important corollaries of Fejér's theorem we find the following facts.

Corollary 1. (Uniqueness) *Suppose $f \in \mathcal{R}(\mathbb{S}^1)$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(\theta) = 0$ whenever f is continuous at θ. In particular, if f is continuous everywhere, then $f \equiv 0$.*

Proof. Since we have $S_N(f)(\theta) = 0$ for all n, all the Cesáro means $\sigma_N(f)(\theta) = 0$ vanish also. The corollary now follows from Theorem 2. □

Corollary 2. **(Weierstrass Approximation on $\mathbb{S}^1$)** *Any $f \in C^0(\mathbb{S}^1)$ can be uniformly approximated by a trigonometric polynomial. That is, for every $\epsilon > 0$, there is a trigonometric polynomial $P = \sum_{n=-N}^{N} c_n e^{inx}$ such that*

$$|f(\theta) - P(\theta)| < \epsilon \quad \textit{for all } \theta \in [-\pi, \pi].$$

Proof. Let $\epsilon > 0$ be given. Since $f \in C^0(\mathbb{S}^1)$ by assumption, we conclude from Theorem 2 that $|f(\theta) - (f * \sigma_N)(\theta)| < \epsilon$ for all θ, provided that N is sufficiently large. As a finite linear combination of exponentials, the trigonometric polynomial $P = (f * F_N)$ has the desired property. □

Abel means. There is another way of "regularizing" the sequence of partial sums $S_N(f)$ in terms of **Abel means** defined as

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} r^{|n|}, \quad \text{with } 0 \leq r < 1.$$

Note that, since $|a_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f| \, d\theta \leq C$, we see that $|A_r(f)(\theta)| \leq C \sum_{n=-\infty}^{\infty} r^{|n|} < \infty$ for $0 \leq r < 1$ by the geometric series formula. Hence, by the Weierstrass M-test, the sum $A_r(f)$ converges uniformly. However, we are interested in taking the limit $r \to 1$, which may not exist. Next, we notice that

$$A_r(f)(\theta) = (f * P_r)(\theta),$$

where $P_r(\theta)$ denotes the **Poisson kernel** indexed by the real parameter $r \in [0, 1)$. It is not difficult to verify that

$$\boxed{P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}}$$

This shows that $P_r(\theta) \geq 0$ and one readily sees that $P_r(\theta)$ is a good kernel in the sense that $\{P_{r_n}\}_{n=1}^{\infty}$ furnishes a family of good kernels for any sequence $r_n \to 1-$.

Thus Theorem 1 yields:

Theorem 3. **(Abel summability)** *Let $f \in \mathcal{R}(\mathbb{S}^1)$. Then the Abel means of the Fourier series of f converge*

$$\lim_{r \to 1} A_r(f)(\theta) = f(\theta),$$

whenever f is continuous at θ. The limit is uniform in θ whenever $f \in C^0(\mathbb{S}^1)$ holds.

A natural application of this theorem is the Dirichlet problem on the unit disc; see Stein $ Shakarchi Section 5.4 for more details. That is, given a function $f : \mathbb{S}^1 \to \mathbb{R}$, we seek a solution $u = u(x, y)$ to

$$\begin{cases} \triangle u = 0, & \text{for } x^2 + y^2 < 1, \\ u = f, & \text{for } x^2 + y^2 = 1. \end{cases}$$

Here $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the so-called **Laplace operator**. Indeed, if a_n denote the Fourier coefficients of f, let us define

$$u(r, \theta) = (f * P_r)(\theta), \quad \text{for } \theta \in [-\pi, \pi] \text{ and } r \in [0, 1),$$

with polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Then $u(r, \theta)$ solves the Dirichlet problem on the unit disc such that

$$\lim_{r \to 1} u(r, \theta) = f(\theta)$$

at every point θ of continuity of f. Moreover, this limit is uniform if $f \in C^0(\mathbb{S}^1)$ and u given above is the only solution with this property.