1 Introduction

My research is in an area known as geometric representation theory. Geometry may be a bit of a misnomer. By geometry, we mean studying the topology of a space X using sheaf theoretic methods. Usually X is an algebraic variety, but the topology we use is either the analytic topology when X is defined over $\mathbb{C}$ or the étale topology when X is defined over $\overline{\mathbb{F}}_q$.

Why sheaves? Functions are such a central concept in mathematics that one cannot escape middle school without getting a taste of them. Grothendieck suggested we replace functions with sheaves, which remember more information about X. For instance, the dimension of the vector space of locally constant functions on X tells us the number of connected components of X, while the category of locally constant *sheaves* also encodes the fundamental group of X. Moreover, operations that can be performed on functions can often be upgraded to operations on sheaves. These operations, sheaf functors, are not exact in general, and so we work in the *derived category*. Also, we restrict our attention to the subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ of *constructible* objects—those that are built out of locally constant sheaves.

Now, suppose that an algebraic group G acts on X. For example, G acts on itself by conjugation or on its Lie algebra by the adjoint action. When X is singular, the structure of these singularities—encoded by the local behavior of perverse sheaves (or sometimes parity sheaves)—often contains deep information about the representation theory of G, or when G is reductive, its Langlands dual group $G^\vee$. Perverse sheaves (and parity sheaves) are certain chain complexes of sheaves living inside $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. We will also work with a subtle variant: the G-equivariant derived category $\mathrm{D}^{\mathrm{b}}_G(X)$. Morever, certain combinatorial 'coincidences' can be explained geometrically and are a result of some equivalence of categories.

For example, let $G = \mathrm{PGL}_n(\mathbb{C})$. Then G acts by conjugation on the variety of $n \times n$-nilpotent matrices $\mathcal{N}$. It has been known for some time that both nilpotent G-orbits and irreducible $\mathbb{C}$-representations of S_n (the symmetric group on n-letters) are both parametrized by partitions of n. This matching of a nilpotent orbit with an irreducible S_n representation can be upgraded to an equivalence of categories known as the Springer correspondence ([14]):

$$\mathrm{Perv}_G(\mathcal{N}) \cong \mathrm{Rep}(S_n).$$

In what follows, I will discuss several such equivalences (of abelian, additive, and triangulated categories) related to my research interests. In Section 2, $\overline{\mathbb{Q}}_\ell$-sheaves on the nilpotent cone are considered. We examine parity sheaves on the affine Grassmannian and their relation to modular representation theory in Section 3. Finally, in Section 4, we study modular tilting perverse sheaves on the affine flag manifold. Throughout, G is a connected, reductive algebraic group defined over $\mathbb{C}$ or $\overline{\mathbb{F}}_q$. Let $k = \overline{\mathbb{F}}_\ell$ with ℓ and q relatively prime. $G^\vee$ is the Langlands dual group defined over k.

2 Lusztig's generalized Springer correspondence

Suppose G is defined over $\overline{\mathbb{F}}_q$ of good characteristic. We consider $\mathcal{N}$, its nilpotent cone with the adjoint G-action. Perverse sheaves on the nilpotent cone are related to Weyl group representations, characters for finite groups of Lie type, and p-adic representation theory.

We denote by $\mathcal{N}_L$ the nilpotent cone for a Levi subgroup L of G and consider the functor of induction from a Levi as defined by Lusztig $\mathcal{I}_L^G : \mathrm{D}^{\mathrm{b}}_L(\mathcal{N}_L) \to \mathrm{D}^{\mathrm{b}}_G(\mathcal{N})$. Lusztig proves that $\mathcal{I}_L^G$ is exact with respect to the perverse t-structure. Almost all simple perverse sheaves on the nilpotent cone can be realized as a summand of an induced perverse sheaf from a Levi subgroup. Those that can't are called *cuspidal*. Thus, to understand perverse sheaves on the nilpotent cone, it is enough to classify the irreducible cuspidal perverse sheaves $\mathbf{c}$ for each Levi L up to G-conjugacy and understand the induced perverse sheaf $\mathbb{A}_{\mathbf{c}} = \mathcal{I}_L^G(\mathbf{c})$. We call $\mathbb{A}_{\mathbf{c}}$ a *Lusztig sheaf*. Let $\mathrm{D}^{\mathrm{b}}_G(\mathcal{N}, \mathbb{A}_{\mathbf{c}})$ be the triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_G(\mathcal{N})$ generated by the simple summands of $\mathbb{A}_{\mathbf{c}}$.

Orthogonal Decomposition and Derived Equivalences

The category $\mathrm{Perv}_G(\mathcal{N})$ is semisimple. However, two irreducible perverse sheaves S and S' may have "geometric" extensions between them, i.e. $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_G(\mathcal{N})}(S, S'[i])$ need not equal 0 for $i > 0$. On the other hand, Lusztig classified the irreducible perverse sheaves on $\mathcal{N}$ in terms of cuspidal data for G ([21]). In fact, $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_G(\mathcal{N})}(S, S'[i]) \neq 0$ implies that S an S' are both summands of the Lusztig sheaf $\mathbb{A}_{\mathbf{c}}$ for exactly one (up to G-conjugacy) cuspidal perverse sheaf $\mathbf{c}$ on the nilpotent cone of a Levi. This is the main theorem of [30], joint work with A. Russell.

Theorem 2.1 (R.–Russell [30])**.** *There is an orthogonal decomposition of the G-equivariant derived category according to Lusztig's classification of cuspidal data* $\mathbf{c}$ *up to G-conjugacy:*

$$\mathrm{D}^{\mathrm{b}}_G(\mathcal{N}) \cong \bigoplus_{\mathbf{c}/\sim} \mathrm{D}^{\mathrm{b}}_G(\mathcal{N}, \mathbb{A}_{\mathbf{c}}).$$

Now we consider the Springer case. We regard the constant sheaf on a point $\underline{\overline{\mathbb{Q}}_\ell}_{\mathrm{pt}}$ as a cuspidal perverse sheaf on the nilpotent cone of the maximal torus T. Induction of this cuspidal gives rise to the Springer sheaf denoted by $\mathbb{A} = \mathcal{I}_T^G(\underline{\overline{\mathbb{Q}}_\ell})$. Note that $\mathrm{End}(\mathbb{A}) \cong \overline{\mathbb{Q}}_\ell[W]$, where W is the Weyl group of G. This ring isomorphism is the main ingredient of Borho–MacPherson's version of Springer's corresponence. Let $\mathcal{A} = \mathrm{Hom}^\bullet(\mathbb{A}, \mathbb{A})$ viewed as a dg-algebra with trivial differential, and let $\mathrm{DG}(\mathcal{A})$ denote the derived category of finitely generated dg-$\mathcal{A}$-modules. In [29], I prove a *derived* version of the Springer correspondence:

Theorem 2.2 (R. [29])**.** *There is an equivalence of triangulated categories* $\mathrm{D}^{\mathrm{b}}_G(\mathcal{N}, \mathbb{A}) \cong \mathrm{DG}(\mathcal{A})$.

To prove this theorem, we employ the powerful technique of mixed geometry typically used in similar equivalences. However, Deligne's category of mixed sheaves $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ (see [8, Section 5] for a definition) is too big, so we need a suitable substitute. Such a substitute is well-studied for flag varieties, but the usual approach as in [10] does not apply in our setting; the topology of nilpotent orbits is more complicated than that of Schubert cells. We first show that Frobenius acts on $\mathrm{Hom}(\mathbb{A}, \mathbb{A}[i])$ by multiplication by $q^{i/2}$ ([29, Lemma 5.2]). This allows a sort-of "motivic" construction of a mixed version of $\mathrm{D}^{\mathrm{b}}_G(\mathcal{N}, \mathbb{A})$ (see [29, Theorem 5.5]), inspired by Achar–Riche's construction of the mixed category for flag varieties in [2].

Fix a cuspidal perverse sheaf $\mathbf{c}$ on $\mathcal{N}_L$ for some Levi L, and let $W(L) = \mathrm{N}_G(L)/L$. Lusztig proves that there is an algebra isomorphism $\mathrm{End}(\mathbb{A}_{\mathbf{c}}) \cong \overline{\mathbb{Q}}_\ell[W(L)]$. Now, we consider the dg-algebra $\mathcal{A}_{\mathbf{c}} = \mathrm{Hom}^\bullet(\mathbb{A}_{\mathbf{c}}, \mathbb{A}_{\mathbf{c}})$. In joint work with Russell, we give the following description of the non-Springer blocks of $\mathrm{D}^{\mathrm{b}}_G(\mathcal{N})$.

Theorem 2.3 (R.–Russell [31])**.** *There is an equivalence of triangulated categories* $\mathrm{D}^{\mathrm{b}}_G(\mathcal{N}, \mathbb{A}_{\mathbf{c}}) \cong \mathrm{DG}(\mathcal{A}_{\mathbf{c}})$.

We follow the procedure of [29]. However, our approach is complicated by the presence of the non-constant local system giving rise to $\mathbb{A}_{\mathbf{c}}$. We are able to show that Frobenius acts on $\mathrm{Hom}(\mathbb{A}_{\mathbf{c}}, \mathbb{A}_{\mathbf{c}}[i])$ by $q^{i/2}$. To do so, we study the cohomology of a generalized Steinberg variety over $\mathbb{F}_q$ in a similar manner to Lusztig's [23, Proposition 4.6]. We are able to simplify the exposition by utilizing Braden's hyperbolic localization [15].

Problem 2.4. *What can be said about the* $G \times \mathbb{G}_m$*-equivariant version of the above picture?*

The categories in question are very similar to the G-equivariant case. For instance, we still have the same collection of simple perverse sheaves. The main difference is that the algebra $\mathrm{Hom}^\bullet(\mathbb{A}_{\mathbf{c}}, \mathbb{A}_{\mathbf{c}})$, taken in the $G \times \mathbb{G}_m$-equivariant derived category, is isomorphic to Lusztig's graded Hecke algebra $\mathbb{H}$. Because of Lusztig's work, the representation theory of the affine Hecke algebra and p-adic groups can be studied using $\mathbb{H}$, whose representations are easier to classify. Bezrukavnikov has suggested the $G \times \mathbb{G}_m$-equivariant case should allow a formulation of Koszul duality for p-adic groups inspired by Vogan's character duality for real groups.

3 Parity sheaves on the affine Grassmannian

A well known result of Arkhipov–Bezrukavnikov–Ginzburg [7] relates constructible sheaves on the affine Grassmannian, coherent sheaves on the (Langlands dual) Springer resolution, and representations of the quantum group

$$\mathrm{D}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{DCoh}^{G^\vee \times \mathbb{G}_m}(\widetilde{\mathcal{N}}) \xleftarrow{\sim} \mathrm{D}^{\mathrm{mix}}(\mathrm{Rep}_{\mathrm{triv}}(U_q)). \tag{3.1}$$

Now I will explain recent progress towards a modular version of this theorem.

Let G be defined over $\mathbb{C}$. In what follows, we consider the k-linear category of perverse sheaves on the affine Grassmannian $\mathcal{G}r$ of G. We stratify $\mathcal{G}r$ in two ways: by spherical orbits and by Iwahori orbits. The $G_{\mathfrak{o}} = G(\mathbb{C}[[t]])$ orbits of $\mathcal{G}r$ are labeled by dominant coweights $\mathbb{X}^+$ for G. The category of spherical perverse sheaves $\mathrm{Perv}_{G_{\mathfrak{o}}}(\mathcal{G}r, k)$ has a convolution $* : \mathrm{Perv}_{G_{\mathfrak{o}}}(\mathcal{G}r, k) \times \mathrm{Perv}_{G_{\mathfrak{o}}}(\mathcal{G}r, k) \to \mathrm{Perv}_{G_{\mathfrak{o}}}(\mathcal{G}r, k)$.

The geometric Satake theorem [20, 28, 17] describes the tensor category of representations of the Langlands dual group $G^\vee$ in terms of spherical perverse sheaves on the affine Grassmannian. Mirković and Vilonen first prove the theorem for integral coefficients and then deduce it for any field k.

Theorem (Geometric Satake, Mirković–Vilonen [28])**.** *We have an equivalence of tensor categories:*

$$\mathcal{S} : (\mathrm{Rep}(G^\vee, k), \otimes) \xrightarrow{\sim} (\mathrm{Perv}_{G_{\mathfrak{o}}}(\mathcal{G}r, k), *).$$

A spherical orbit labeled by $\lambda \in \mathbb{X}^+$ is a union of $|W\lambda|$-many Iwahori orbits. This stratifies $\mathcal{G}r$ into cells indexed by all coweights $\mathbb{X}$ for G. We denote an Iwahori orbit by $I\lambda$ for $\lambda \in \mathbb{X}$. In [16], Finkelberg–Mirković suggest another equivalence relating Iwahori constructible perverse sheaves on $\mathcal{G}r$ to the block of $\mathrm{Rep}(G^\vee, k)$ containing the trivial representation, denoted $\mathrm{Rep}_{\mathrm{triv}}(G^\vee, k)$.

Conjecture (Finkelberg–Mirković, [16]). *There is an equivalence of abelian categories*

$$\mathcal{P} : \mathrm{Perv}_{(I)}(\mathcal{G}r, k) \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{triv}}(G^\vee, k).$$

Let $U_q = U_q(\mathfrak{g}^\vee)$ be the quantum group at an odd root of unity (of order bigger than the Coxeter number, coprime to 3 in type G_2). Arkhipov–Bezrukavnikov–Ginzburg [7] proved the following characteristic 0 version of the Finkelberg–Mirković Conjecture:

$$\mathrm{Perv}_{(I)}(\mathcal{G}r, \mathbb{C}) \cong \mathrm{Rep}_{\mathrm{triv}}(U_q).$$

The main idea of their proof is relating both categories to coherent sheaves on the Springer resolution $\widetilde{\mathcal{N}}$ for $G^\vee$ as in equivalence (3.1).

3.1 Progress in positive characteristic

In recent years, Soergel's bimodules have inspired the discovery of a remarkable collection of objects in the constructible derived category. These are Juteau–Mautner–Williamson's parity sheaves (definition in our setting below). Parity sheaves exhibit many of the properties that make pointwise pure intersection cohomology complexes such a powerful tool in characteristic 0 representation theory.

Theorem (Juteau–Mautner–Williamson [18]). *For each $\lambda \in \mathbb{X}$, there is a unique indecomposable $\mathcal{E}_\lambda \in \mathrm{D}_{(I)}(\mathcal{G}r, k)$ such that the support of $\mathcal{E}_\lambda$ is contained within $\overline{I\lambda}$, the stalks and costalks of $\mathcal{E}_\lambda$ are concentrated in degrees congruent to $\dim(I\lambda)$ mod 2, and $\mathcal{E}_\lambda|_{I\lambda} = \underline{k}[\dim I\lambda]$.*

Definition 3.1. [18] The chain complex $\mathcal{E} \in \mathrm{D}_{(I)}(\mathcal{G}r, k)$ is called a parity sheaf if $\mathcal{E} \cong \bigoplus_i \mathcal{E}_{\lambda_i}[d_i]$ for some finite collection $\lambda_i \in \mathbb{X}$ and $d_i \in \mathbb{Z}$. Let $\mathrm{Parity}_{(I)}(\mathcal{G}r)$ be the additive category of parity sheaves. (Note: The definition of parity sheaves does not require an affine stratification, but they may not exist in general.)

Juteau–Mautner–Williamson show that the parity vanishing of stalks (and costalks) geometrically characterizes the representation theoretic property of being a *tilting* representaiton. JMW must impose some mild assumptions on $\mathrm{char}(k)$. However, Mautner–Riche recently improved these bounds in [26] to good characteristic for $G^\vee$.

Theorem ([19], [26]). *Suppose $\mathrm{char}(k)$ is good. For $\lambda \in \mathbb{X}^+$, the indecomposable parity sheaf $\mathcal{E}_\lambda$ is perverse and corresponds under geometric Satake to the indecomposable tilting $G^\vee$ representation of highest weight λ.*

JMW's theorem turns out to be equivalent to a conjecture of Mirković–Vilonen from 2000 [27] (suitably modified by Juteau), recently proven in 2013 in my joint work with P. Achar:

Theorem 3.2 (Achar–R. [4]). *The $\mathbb{Z}$-stalks of spherical intersection cohomology sheaves on $\mathcal{G}r$ have no p-torsion for p a good prime.*

A key component of our proof (see [4]) is an extension of JMW's theorem. We show it holds when the derived group of $G^\vee$ is simply connected, and its Lie algebra admits a nondegenerate $G^\vee$-invariant bilinear form which we assume from now on.

Theorem 3.3 (Achar–R. [4])**.** *$\mathcal{S}$ extends to an equivalence of additive categories*

$$\mathrm{Parity}_{(G_o)}(\mathcal{G}r) \cong \mathrm{Tilt}(\mathcal{N}).$$

Here, $\mathcal{N}$ is the nilpotent cone for $G^\vee$ and $\mathrm{Tilt}(\mathcal{N})$ is the category of tilting perverse coherent sheaves on $\mathcal{N}$. To prove this theorem, we interpret Ext between two parity sheaves in a manner similar to Ginzburg.

Another application of parity sheaves is Achar–Riche's definition of the mixed modular derived category of sheaves on a flag variety (see [3]). This technology gives one interpretation of a modular version of equivalence (3.1).

Theorem 3.4 (Achar–R. [5])**.** *Assume* $\mathrm{char}(k)$ *is good for* $G^\vee$*. We have an equivalence of triangulated categories:*

$$\mathcal{P} : \mathrm{D}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, k) \xrightarrow{\sim} \mathrm{DCoh}^{G^\vee \times \mathbb{G}_m}(\widetilde{\mathcal{N}})$$

such that $\mathcal{P}(F\langle 1\rangle) \cong \mathcal{P}(F)\langle -1\rangle[1]$*. Furthermore,* $\mathcal{P}$ *is compatible with geometric Satake, i.e. for* $V \in \mathrm{Rep}(G^\vee, k)$*,* $\mathcal{P}(F * \mathcal{S}(V)) \cong \mathcal{P}(F) \otimes V$*.*

Our proof of Theorem 3.4 requires stronger restrictions on the characteristic of k, but [26] implies the argument is valid in good characteristic.

The category $\mathrm{DCoh}^{G^\vee \times \mathbb{G}_m}(\widetilde{\mathcal{N}})$ has a t-structure defined by Bezrukavnikov in [11]. The heart of this t-structure is denoted $\mathrm{ExCoh}(\widetilde{\mathcal{N}})$, and objects therein are called exotic coherent sheaves. The category $\mathrm{ExCoh}(\widetilde{\mathcal{N}})$ is a graded highest weight category ([25]) with indecomposable tilting objects in bijection with $\mathbb{X} \times \mathbb{Z}$.

Corollary 3.5 (Achar–R. [5])**.** *$\mathcal{P}$ restricts to an equivalence of additive categories:*

$$\mathrm{Parity}_{(I)}(\mathcal{G}r) \cong \mathrm{Tilt}(\widetilde{\mathcal{N}}).$$

In our proof of Theorem 3.4, we rely heavily on so-called *Wakimoto sheaves*. These are the constructible analogue of line bundles on $\widetilde{\mathcal{N}}$. We prove that Wakimoto sheaves lift to the mixed category. We then construct two functors: the correct functor which is triangulated, and another naive functor which is only additive. However, the naive functor has the advantage that we can compute the output. We then use weight arguments to show that the images of these functors match for nice enough objects. One of the reasons we expected this method to work is that we knew that tilting exotic coherent sheaves, a priori chain complexes of coherent sheaves, should be coherent. This also becomes a consequence of our theorem and the study of the transport of the exotic t-structure on $\mathrm{DCoh}^{G^\vee \times \mathbb{G}_m}(\widetilde{\mathcal{N}})$ to $\mathrm{D}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, k)$.

Remark 3.6. In [26], Mautner–Riche prove Theorem 3.4 by very different methods. They utilize braid group actions on both categories. They first prove Corollary 3.5 and obtain Theorem 3.4 as a consequence. Note that their proof does *not* rely on geometric Satake and has fewer restrictions on characteristic of k. However, it is not clear by their methods that the equivalence is compatible with geometric Satake.

3.2 Some Related Problems

Problem 3.7. *For* $\lambda \in \mathbb{X}$*, is the indecomposable parity sheaf* $\mathcal{E}_\lambda$ *perverse?*

A positive solution to the above problem would give an interesting class of representations under the conjectural equivalence of Finkelberg–Mirković. We note that this is known to hold for type A_2. This follows from Achar's calculations of exotic coherent sheaves in [1] and the equivalence in Corollary 3.5. Alternatively, one can check this directly on the constructible side since each $\mathcal{E}_\lambda$ is either spherical or the convolution of a skyscraper on a non-identity component of $\mathcal{G}r$ with a spherical perverse parity sheaf.

Problem 3.8. *Can one deduce that* $\mathrm{Perv}_{G_\mathrm{o}}(\mathcal{G}r, k)$ *is a highest weight category without appealing to geometric Satake?*

If so, then Mautner–Riche's recent study of the equivalence (3.4) would give another proof of the equivalence of abelian categories $\mathrm{Rep}(G^\vee, k) \cong \mathrm{Perv}_{G_\mathrm{o}}(\mathcal{G}r, k)$, without appealing to Tannakian formalism. In joint work with R. Bezrukavnikov and D. Gaitsgory, I have begun to investigate an equivalence conjectured by Gaitsgory between the Satake category and an Iwahori-Whittaker model. Since the Iwahori-Whittaker category is easily seen to be highest weight, the proof may shed some light on the highest weight structure of $\mathrm{Perv}_{G_\mathrm{o}}(\mathcal{G}r, k)$.

Problem 3.9. *Is there a natural degrading functor* $\mathrm{D}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, k) \to \mathrm{D}_{(I)}(\mathcal{G}r, k)$*?*

Achar–Riche prove existence of a degrading functor for finite dimensional flag varieties. An unmixed version of Theorem 3.4 becomes feasible with a positive solution to Problem 3.9 which would be considerable progress towards a proof of the Finkelberg–Mirković Conjecture.

I. Losev and I have begun investigating an alternative proof of the Finkelberg–Mirković Conjecture in type A which relies on uniqueness of categorification of $\mathfrak{sl}_\infty$-actions.

4 Antispherical module and the dual the Springer resolution

Let $\mathcal{F}\ell$ be the affine flag manifold for G defined over $\overline{\mathbb{F}}_q$. Over $\overline{\mathbb{F}}_q$, $\mathbb{A}^1$ supports interesting local systems such as the Artin–Schreier sheaf. This allows a definition of the Iwahori-Whittaker category and perverse sheaves therein: $\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}\ell, k) \subset \mathrm{D}^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}\ell, k)$. Sheaves in $\mathrm{D}^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}\ell, k)$ are constructible with respect to I_- (opposite Iwahori) orbits and equivariant with repect to an additive character of I^u_-, the pro-unipotent radical of I_-. Note that while I and I_--orbits of $\mathcal{F}\ell$ are indexed by the extended affine Weyl group W_{aff} of G, the irreducible perverse sheaves in $\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}\ell, k)$ are indexed by $\mathbb{X}$, the set of coweights of G. The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}\ell, k)$ is one realization of the antispherical module for the affine Hecke category.

Let $\widetilde{\mathcal{N}}$ be the Springer resolution for $G^\vee$ defined over $\mathbb{C}$. Arkhipov–Bezrukavnikov prove a derived equivalence relating the Iwahori–Whittaker realization of the antispherical module and the derived category of coherent sheaves on $\widetilde{\mathcal{N}}$ with characteristic 0 coefficients:

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}\ell, \overline{\mathbb{Q}}_\ell) \cong \mathrm{DCoh}^{G^\vee}(\widetilde{\mathcal{N}}). \tag{4.1}$$

Now assume that $G^\vee$ and its Springer resolution $\widetilde{\mathcal{N}}$ are defined over k. I expect the modular analogue of equivalence (4.1) to hold:

Conjecture 4.1. *Suppose that* $\mathrm{char}(k)$ *is good for* G*. There is an equivalence of derived categories* $\mathrm{D}^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}\ell, k) \cong \mathrm{DCoh}^{G^\vee}(\widetilde{\mathcal{N}})$ *that sends the perverse t-structure to the exotic t-structure.*

However, my approach is very different from that of [6]. Both categories are equivalent to the derived category of a highest weight category. Hence, it is enough to identify the additive category of tilting objects in the two categories. To do so, we use an approach inspired by Soergel. We consider 'maximal' quotients of both categories that still contains the relevant information about the tilting objects, and then hope that the quotients are easy to identify.

The quotient we consider on the coherent side is given by restriction to the regular elements $\widetilde{\mathcal{N}}_{\mathrm{reg}}$. (To see this as a quotient, we must work with Bezrukavnikov's exotic t-stucture as defined in [11].)

Proposition 4.2. *The restriction functor* $\mathrm{DCoh}^{G^\vee}(\widetilde{\mathcal{N}}) \to \mathrm{DCoh}^{G^\vee}(\widetilde{\mathcal{N}}_{\mathrm{reg}})$ *is fully faithful on tilting exotic coherent sheaves.*

To prove this statement, I verify a conjecture of Achar in [1, 4.13] which concerns the socle of standard exotic sheaves using the braid group action and recent results of [25] concerning the exotic t-structure.

On the constructible side, we let $\mathcal{M}$ be the triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}\ell, k)$ generated by irreducible perverse sheaves such that the dimension of support is greater than 1. One can use techniques of [9] to prove the following.

Proposition 4.3. *The quotient functor* $\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}\ell) \to \mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}\ell)/\mathcal{M}$ *is fully faithful on tilting perverse sheaves.*

Now, we must relate the two quotients so that the images of tilting objects match. Using Tannakian formalism, one can deduce that the quotient category $\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}\ell)/\mathcal{M}$ is equivalent to $\mathrm{Rep}(H)$ for some H, an algebraic subgroup of $G^\vee$. We must show that H is the centralizer of a regular nilpotent in $G^\vee$.

Problem 4.4. *Identify the subgroup H of $G^\vee$ such that the quotient category* $\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}\ell)/\mathcal{M}$ *is equivalent to Rep(H).*

Let I^u be the pro-unipotent radical of the Iwahori I. Now we consider a category of perverse sheaves on the extended affine flag variety $\widetilde{\mathcal{F}\ell} := G(\mathbb{C}((t)))/I^u$. Let $\mathrm{Perv}(\widetilde{\mathcal{F}\ell}, k)$ denote the category of I^u-equivariant, $T \times T$-monodromic perverse sheaves on $\widetilde{\mathcal{F}\ell}$.

Problem 4.5. *Study the analogue of the quotient functor in Proposition 4.3 for* $\mathrm{Perv}(\widetilde{\mathcal{F}\ell})$*. In particular, show it is fully-faithful on tilting perverse sheaves, and identify the quotient category.*

Bezrukavnikov–Yun [13, Prop. 4.5.7] prove fully-faithfulness in the setting of mixed ℓ-adic sheaves using a similar strategy as in [9]. However, subtleties arise because one must work with a certain completion of the category to allow "free-monodromic" tilting sheaves (See Yun's [13, Appendix A]).

On the coherent side, we consider the Langlands dual Steinberg variety $\mathrm{St} = \tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee$. The category $\mathrm{DCoh}^{G^\vee}(\mathrm{St})$ also has an exotic t-structure defined using braid group actions.

Question 4.6. *Is the functor of restriction from* $\mathrm{DCoh}^{G^\vee}(\mathrm{St})$ *to the Kostant slice fully-faithful on tilting objects?*

Understanding the above problems would likely lead to a proof of a modular version of Bezrukavnikov's theorem relating two realizations of the affine Hecke algebra and a different proof of the characteristic 0 theorem.

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