The left and right homotopy relations

We recall that a coproduct of two objects \( A \) and \( B \) in a category \( C \) is an object \( A \sqcup B \) together with two maps \( \text{in}_1 : A \to A \sqcup B \) and \( \text{in}_2 : B \to A \sqcup B \) such that, for every pair of maps \( f : A \to C \) and \( g : B \to C \), there exists a unique map

\[
f + g : A \sqcup B \to C
\]

such that \( f = (f + g) \circ \text{in}_1 \) and \( g = (f + g) \circ \text{in}_2 \). If both \( A \sqcup B \) and \( A \sqcup B' \) are coproducts of \( A \) and \( B \), then the maps \( \text{in}_1 + \text{in}_2' : A \sqcup B \to A \sqcup B' \) and \( \text{in}_1 + \text{in}_2' : A \sqcup B' \to A \sqcup B \) are isomorphisms and each others inverses. The map \( \nabla = \text{id} + \text{id} : A \sqcup A \to A \) is called the fold map. Dually, a product of two objects \( A \) and \( B \) in a category \( C \) is an object \( A \times B \) together with two maps \( \text{pr}_1 : A \times B \to A \) and \( \text{pr}_2 : A \times B \to B \) such that, for every pair of maps \( f : C \to A \) and \( g : C \to B \), there exists a unique map

\[
(f, g) : C \to A \times B
\]

such that \( f = \text{pr}_1 \circ (f, g) \) and \( g = \text{pr}_2 \circ (f, g) \). If both \( A \times B \) and \( A \times' B \) are products of \( A \) and \( B \), then the maps \( (\text{pr}_1, \text{pr}_2) : A \times B \to A \times' B \) and \( (\text{pr}_1', \text{pr}_2') : A \times' B \to A \times B \) are isomorphisms and each others inverses. The map \( \Delta = (\text{id}, \text{id}) : A \to A \times A \) is called the diagonal map.

**Definition** Let \( C \) be a model category, and let \( f : A \to B \) and \( g : A \to B \) be two maps. A cylinder object for \( A \) is a commutative diagram

![Cylinder Diagram](image)

and a left homotopy from \( f \) to \( g \) is a commutative diagram

![Left Homotopy Diagram](image)

If a left homotopy from \( f \) to \( g \) exists, we say that \( f \) and \( g \) are left homotopic and write \( f \sim^l g \). Dually, a path object for \( B \) is a commutative diagram

![Path Diagram](image)
and a right homotopy from $f$ to $g$ is a commutative diagram

$$
\begin{array}{c}
\text{Path}(B) \\
\downarrow \downarrow \downarrow \downarrow \\
A \xrightarrow{(f,g)} B \times B.
\end{array}
$$

If a right homotopy from $f$ to $g$ exists, we say that $f$ and $g$ are right homotopic and write $f \sim^r g$. If both a left and a right homotopy from $f$ to $g$ exist, we say that $f$ and $g$ are homotopic and write $f \sim g$.

The homotopy relations $\sim^l$, $\sim^r$, and $\sim$ are relations on the set $\text{Hom}_C(A, B)$ of maps from $A$ to $B$ in $C$. But, in general, they are not equivalence relations. We write $\text{Hom}_C(A, B)/\sim^l$, $\text{Hom}_C(A, B)/\sim^r$, and $\text{Hom}_C(A, B)/\sim$ for the sets of equivalence classes for the equivalence relations on the set $\text{Hom}_C(A, B)$ generated by the relations $\sim^l$, $\sim^r$, and $\sim$, respectively.