18.727: Problem Set 9

Due: 5/9/01

1. Let $X$ be an irreducible projective scheme over a field $k$. Show that $H^0(X, \mathcal{O}_X)$ is a finite extension field of $k$.

2. Let $A$ be a noetherian ring, let $X = \text{Spec} A$, and let $M$ and $N$ be finitely generated $A$-modules.

   (i) Show that for all $q \geq 0$, the groups $\text{Ext}^q_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$ and $\text{Ext}^q_A(M, N)$ are naturally isomorphic.

   (ii) Show that for all $q \geq 0$, the sheaves $\text{Ext}^q_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$ and $\text{Ext}^q_A(M, N)$ are naturally isomorphic.

3. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $M$ is of finite type if there exists a covering $\{U_i \to X\}_{i \in I}$, and for all $i \in I$, a surjection from a finite direct sum of copies of $\mathcal{O}_X|_{U_i}$ onto $M|_{U_i}$. It is coherent if it is of finite type and if for every open subset $U \subset X$ and every homomorphism $f: (\mathcal{O}_X|_U)^n \to M|_U$, the kernel of $f$ is of finite type on $(U, \mathcal{O}_X|_U)$. Show:

   (i) If $M$ is a coherent $\mathcal{O}_X$-module and $N \subset M$ is a sub-$\mathcal{O}_X$-module of finite type, then $N$ is coherent.

   (ii) If two out of three $\mathcal{O}_X$-modules in an exact sequence of $\mathcal{O}_X$-modules

   \[ 0 \to M' \to M \to M'' \to 0 \]

   are coherent, then so is the third.

   (iii) If $M$ and $N$ are coherent $\mathcal{O}_X$-modules, then so are $M \otimes_{\mathcal{O}_X} N$ and $\text{Hom}_{\mathcal{O}_X}(M, N)$. 