1. Let \((X, A)\) be a ringed space, and let \(\mathcal{U} = \{U_i \to X\}_{i \in I}\) be a family of open subsets. We define \(A_\mathcal{U}\) to be the presheaf of \(A\)-modules given by

\[
A_\mathcal{U}(V) = \begin{cases} 
A(V), & \text{if } V \subset U_i, \text{ some } i \in I, \\
0, & \text{else}.
\end{cases}
\]

(If the family \(\mathcal{U}\) has only one element \(U \to X\), we write \(A_U\) instead of \(A_\mathcal{U}\).) We choose a well-ordering of the index set \(I\) and consider the complex

\[
\bigoplus_{i_0} A_{U_{i_0}} \leftarrow \bigoplus_{i_0 < i_1} A_{U_{i_0} \cap U_{i_1}} \leftarrow \bigoplus_{i_0 < i_1 < i_2} A_{U_{i_0} \cap U_{i_1} \cap U_{i_2}} \leftarrow \cdots .
\]

(i) Show that this is a resolution of the presheaf of \(A\)-modules \(A_\mathcal{U}\) by projective presheaves of \(A\)-modules.

(Hint: A sequence of presheaves of \(A\)-modules, by definition, is exact if and only if for all \(V \subset X\) open, the sequence of \(A(V)\)-modules obtained by taking sections over \(V\) is exact.)

(ii) Conclude that if \(\mathcal{U}\) is a covering, the Čech cohomology \(\check{H}^*(\mathcal{U}, M)\) is canonically isomorphic to the cohomology of the complex

\[
\prod_{i_0} M(U_{i_0}) \to \prod_{i_0 < i_1} M(U_{i_0} \cap U_{i_1}) \to \prod_{i_0 < i_1 < i_2} M(U_{i_0} \cap U_{i_1} \cap U_{i_2}) \to \cdots .
\]

This is called the complex of alternating Čech cochains of \(M\) with respect to the covering \(\mathcal{U}\).

(Warning: The above is not true, in general, for a ringed topos.)

2. Prove that every non-empty quasi-compact scheme has a closed point.

3. Let \(X\) be a quasi-compact scheme and suppose that for all quasi-coherent \(\mathcal{O}_X\)-modules \(M\) and for all \(q > 0\), \(H^q(X, M)\) is zero.

(i) Let \(x \in X\) is a closed point, let \(x \in U \subset X\) be an affine open neighborhood, and let \(Y = X \setminus U\) be the complement considered as a closed subscheme of \(X\) with the reduced induced scheme structure. Show that there exists \(f \in \Gamma(X, \mathcal{J}_Y)\) such that \(f(x) = 1 \in k(x)\).

(Hint: Use the exact sequence \(0 \to \mathcal{J}_{Y | U \mid x} \to \mathcal{J}_Y \to i_* k(x) \to 0\).)

(ii) Conclude that for every closed point \(x \in X\), there exists \(x \in \Gamma(X, \mathcal{O}_X)\) such that \(X_f = \{x \in X \mid f(x) \neq 0\}\) is an affine open neighborhood of \(x\).

(iii) Show that \(X\) is affine.

(Hint: It suffices to find \(f_1, \ldots , f_n \in \Gamma(X, \mathcal{O}_X)\) such that all \(X_{f_i}\) are affine and such that \((f_1, \ldots , f_n) = \Gamma(X, \mathcal{O}_X)\).)