18.727: Problem Set 6

Due: 4/11/01

1. In an abelian category which both has enough projectives and injectives, show that there is a canonical isomorphism

\[(R^i \text{Hom}(M, -))(N) \cong (R^i \text{Hom}(-, N))(M).\]

\((\text{Hint: let } M \leftarrow P, (\text{resp. } N \rightarrow I) \text{ be a projective (resp. injective) resolution and consider the spectral sequences associated with the double complex } \text{Hom}(P, I)).\)

2. Show that if an additive functor has an exact left adjoint then it preserves injectives.

3. Let \(C^\cdot\) be a (cochain) complex with a descending filtration

\[C^\cdot = \text{Fil}_0^p C^\cdot \supset \text{Fil}_1^p C^\cdot \supset \cdots \supset \text{Fil}_p^p C^\cdot \supset \cdots\]

and assume that \(H^s(\text{Fil}_p^p C^\cdot)\) vanishes, if \(s < p\). For every \(p \geq 0\), we have the long-exact homology sequence

\[\cdots \rightarrow H^s(\text{Fil}_{p+1}^p C^\cdot) \xrightarrow{\delta} H^s(\text{Fil}_p^p C^\cdot) \xrightarrow{\delta} H^{s+1}(\text{Fil}_{p+1}^p C^\cdot) \rightarrow \cdots\]

We let

\[E^{p,q}_r = \frac{k^{-1}(\text{im}(H^{p+q+1}(\text{Fil}_{p+r}^p C^\cdot) \xrightarrow{\delta} H^{p+q}(\text{Fil}_{p+1}^p C^\cdot)))}{j(\ker(H^{p+q}(\text{Fil}_{p}^p C^\cdot) \xrightarrow{\delta} H^{p+q}(\text{Fil}_{p-r}^p C^\cdot)))},\]

and define the \(r\)th differential

\[d_r : E^{p,q}_r \rightarrow E^{p+r,q-(r-1)}_r\]

as follows: Given \(x \in E^{p,q}_r\), we first choose \(\tilde{x} \in H^{p+q}(\text{Fil}_{p+r}^p C^\cdot)\) which represents \(x\). Then \(k(\tilde{x}) \in H^{p+q+1}(\text{Fil}_{p+r}^p C^\cdot)\) only depends on \(x\), and by the definition of \(E^{p,q}_r\), we can choose \(\tilde{y} \in H^{p+q+1}(\text{Fil}_{p+r}^p C^\cdot)\) such that \(i^{r-1}(\tilde{y}) = k(\tilde{x})\). Then, by definition, \(d_r x \in E^{p+r,q-(r-1)}_r\) is the class represented by \(j(\tilde{y}) \in H^{p+q+1}(\text{Fil}_{p+r}^p C^\cdot)\).

(i) Show that \(E^{p,q}_{p+1}\) is isomorphic to the cohomology of \(E^{p-r,q+(r-1)}_{p} \xrightarrow{d_r} E^{p,q}_{p} \xrightarrow{d_r} E^{p+r,q-(r-1)}_{p}\).

(ii) Show that \(E^{p,q}_{\infty} \cong \text{gr}^p H^{p+q}(C^\cdot)\).

4. In an abelian category which has enough injectives, let \(P^\cdot\) be a chain complex (i.e. maps go \(P_0 \leftarrow P_1 \leftarrow \ldots\)) of projective objects. If \(M\) is any object then \(\text{Hom}(P^\cdot, M)\) is a cochain complex. Show that there is a spectral sequence

\[E_2^{s,t} = \text{Ext}^s(H_t(P^\cdot), M) \Rightarrow H^{s+t}(\text{Hom}(P^\cdot, M)).\]

This is called the universal coefficient spectral sequence.