18.727: Problem Set 2

Due: 2/28/01

The purpose of this problem set is to construct the normalization of an integral scheme. We accomplish this in a series of steps.

1. Let $X$ be an irreducible topological space. Show that every non-empty open subset $U \subset X$ is dense and that $U$ itself is an irreducible topological space.

2. A scheme $X$ is integral if for every open subset $U \subset X$, the ring $\Gamma(U, \mathcal{O}_X)$ is an integral domain. Show that a scheme is integral if and only if it is reduced and irreducible.

3. Let $f: X \to X'$ be a morphism between integral schemes. Show that the following are equivalent:
   (i) the image $f(X) \subset X'$ is dense;
   (ii) if $U \subset X$ and $U' \subset X'$ are affine open subsets such that $f(U) \subset U'$, then the composite ring homomorphism
   $\Gamma(U', \mathcal{O}_{X'}) \to \Gamma(f^{-1}(U'), \mathcal{O}_{X'}) \to \Gamma(U, \mathcal{O}_X)$
   is injective.

   Such a map $f$ is called dominant. (Hint: to show that (i) implies (ii), show that if $f$ is in the kernel, then for all $x' \in U'$, $f(x') = 0$. Conclude that $f = 0$. To show that (ii) implies (i) show that $f$ maps the generic point of $U$ to the generic point of $U'$.)

4. An integral scheme $X$ is normal if for every affine open subset $U \subset X$, the ring $\Gamma(U, \mathcal{O}_X)$ is integrally closed (in its quotient field). Let $X$ be an integral scheme. A dominant morphism $f: \tilde{X} \to X$ with $\tilde{X}$ normal is called a normalization of $X$ if it is universal with this property, i.e. if every dominant morphism $g: Z \to X$ with $Z$ normal factors uniquely through $f$.

   (i) Suppose that $X = \text{Spec } R$ is affine, and let $R \to \tilde{R}$ be the canonical map from $R$ to the integral closure of the ring $R$ in its quotient field. Show that the induced map $\text{Spec } \tilde{R} \to \text{Spec } R$ is a normalization.

   (ii) Suppose that $\tilde{X} \to X$ is a normalization and let $U \subset X$ be an open subset. Show that the projection $U \times_X \tilde{X} \to U$ is a normalization. (Hint: Use the universal properties.)

   (iii) Show that every integral scheme $X$ has a normalization $\tilde{X} \to X$. (Hint: Let $\{U_i\}$ be an affine open cover of $X$. Use (i) to find a normalization $\tilde{U}_i \to U_i$. Use (ii) to show that the schemes $\tilde{U}_i$ can be glued together to give $\tilde{X}$. Show that the maps $\tilde{U}_i \to U_i$ glue to give $\tilde{X} \to X$.)