18.727: Problem Set 1

Due: No, but you should do it anyway. Really!

1. (The following is proved in Mumford, but try to do it yourself.) Let $X$ be the prime spectrum of a ring. Show that the closure $\{x\}$ of the one-point set $\{x\} \subset X$ is an irreducible closed subset, i.e. that it cannot be written as the union of two proper closed subsets. Show that $x$ is a generic point of $\{x\}$, i.e. that the only closed subset of $\{x\}$ which contains $x$ is the whole set. Show that every irreducible closed subset of $X$ is of the form $\{x\}$ and that $x$ is its unique generic point. Conclude that the assignment $x \mapsto \{x\}$ defines a one-to-one correspondence between the points of $X$ and the irreducible closed subsets of $X$.

2. Let $X$ be a space, let $x \in X$ be a point, and let $E$ be a set. We define the skyscraper sheaf $i_x^* E$ on $X$ by assigning to $U \subset X$ the set $E$ if $x \in U$, and a one-point set, if $x \not\in U$. Show that $i_x^* E$ is indeed a sheaf. Then show that the functor $E \mapsto i_x^* E$ is right adjoint to the functor $F \mapsto F_x$, which to a sheaf of sets $F$ on $X$ assigns the stalk in the point $x$. (This means that there is natural one-to-one correspondence between maps of sets $F_x \rightarrow E$ and maps of sheaves $F \rightarrow i_x^* E$.) Finally, show that the stalk of $i_x^* E$ at $x' \in X$ is $E$, if $x' \in \{x\}$, and a one-point set, otherwise.

3. Show that Spec $R$ is connected if and only if $R$ does not contain non-trivial idempotents. (A non-trivial idempotent is an element $e \neq 1$ such that $e^2 = e$.)

4. Let $(X, \mathcal{O}_X)$ be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define $X_f$ to be the set of points $x \in X$ such that $f(x) \neq 0 \in k(x)$.

(a) If $U$ is an affine open of $X$, and if $f|_U$ is the image of $f$ in $\Gamma(U, \mathcal{O}_X)$, show that $U \cap X_f = D(f|_U)$. Conclude that $X_f \subset X$ is open.

(b) Assume that $X$ is quasi-compact. Suppose that the restriction of $a \in \Gamma(X, \mathcal{O}_X)$ to $\Gamma(X_f, \mathcal{O}_X)$ is zero. Show that for some $n \geq 0$, $f^n a = 0$.

(c) Assume in addition that $X$ is quasi-separated, i.e. that the intersection of two affine open subsets $U, V \subset X$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_X)$. Show that for some $n \geq 0$, $f^n b$ is the restriction of an element of $\Gamma(X, \mathcal{O}_X)$.

(c) Conclude that if $X$ is quasi-compact and quasi-separated, then the restriction induces an isomorphism

$$\Gamma(X, \mathcal{O}_X)_f \cong \Gamma(X_f, \mathcal{O}_X).$$