STABLE TOPOLOGICAL CYCLIC HOMOLOGY IS TOPOLOGICAL HOCHSCHILD HOMOLOGY

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1. Introduction

1.1. Topological cyclic homology is the codomain of the cyclotomic trace from algebraic $K$-theory

$$\text{trc}: K(L) \rightarrow TC(L).$$

It was defined in [2] but for our purpose the exposition in [6] is more convenient. The cyclotomic trace is conjectured to induce a homotopy equivalence after $p$-completion for a certain class of rings including the rings of algebraic integers in local fields of positive residue characteristic $p$. We refer to [11] for a detailed discussion of conjectures and results in this direction.

Recently B. Dundas and R. McCarthy have proven that the stabilization of algebraic $K$-theory is naturally equivalent to topological Hochschild homology,

$$K^S(R; M) \simeq T(R; M)$$

for any simplicial ring $R$ and any simplicial $R$-module $M$, cf. [4]. We note that both functors are defined for pairs $(L; P)$ where $L$ is a functor with smash product and $P$ is an $L$-bimodule; cf. [12]. An outline of a proof in this setting and by quite different methods, has been given by R. Schwänzl, R. Staffelt, and F. Waldhausen. Hence the following result is a necessary condition for the conjecture mentioned above to hold.

Theorem. Let $L$ be a functor with smash product and $P$ an $L$-bimodule. Then there is a natural weak equivalence, $TC^S(L; P)^p_n \simeq T(L; P)^p_n$.

It is not surprising that we have to $p$-complete in the case of $TC$ since the cyclotomic trace is really an invariant of the $p$-completion of algebraic $K$-theory, cf. 1.4 below. The rest of this paragraph recalls cyclotomic spectra, topological Hochschild homology, topological cyclic homology and stabilization. In paragraph 2 we decompose topological Hochschild homology of a split extension of $FSP$’s and approximate $TC$ in a stable range. Finally in paragraph 3 we study free cyclic objects and use them to prove the theorem.

Throughout $G$ denotes the circle group, equivalence means weak homotopy equivalence and a $G$-equivalence is a $G$-map which induces an equivalence of $H$-fixed sets for any closed subgroup $H \leq G$.

I want to thank my adviser Ib Madsen for much help and guidance in the preparation of this paper as well as in my graduate studies as a whole. Part of this work was done during a stay at the University of Bielefeld and it is a pleasure to thank the university and in particular Friedhelm Waldhausen for their hospitality. I also want to thank him and John Klein for many enlightening discussions.

1.2. Let $L$ be an $FSP$ and let $P$ be an $L$-bimodule. Then $\text{THH}(L; P)$, is the simplicial space with $k$-simplices

$$\text{holim}_{k+1} F(S^{i_0} \wedge \ldots \wedge S^{i_k}, P(S^{i_0}) \wedge L(S^{i_1}) \wedge \ldots \wedge L(S^{i_k}))$$

The author was supported by the Danish Natural Science Research Council

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and Hochschild-type structure maps, cf. [12], and $\text{THH}(L; P)$ is its realization. When $P = L$, considered as an $L$-bimodule in the obvious way, $\text{THH}(L; L)$ is a cyclic space so $\text{THH}(L; L)$ has a $G$-action. In both cases we use a thick realization to ensure that we get the right homotopy type, cf. the appendix. More generally if $X$ is some space we let $\text{THH}(L; P; X)$, be the simplicial space

$$\text{holim}_{i+k+1} F(S^0 \wedge \ldots \wedge S^k, P(S^0) \wedge L(S^0) \wedge \ldots \wedge L(S^k) \wedge X),$$

where $X$ acts as a dummy for the simplicial structure maps. If $X$ has a $G$-action then $\text{THH}(L; P; X)$ becomes a $G$-space and $\text{THH}(L; L; X)$ a $G \times G$-space. We shall view the latter as a $G$-space via the diagonal map $\Delta: G \to G \times G$ and then denote it $\text{THH}(L; X)$.

We define a $G$-prespectrum $t(L; P)$ in the sense of [9] whose $0$'th space is $\text{THH}(L; P)$. Let $V$ be any orthogonal $G$-representation, or more precisely, any f.d. sub inner product space of a fixed ‘complete $G$-universe’ $U$. Then

$$t(L; P)(V) = \text{THH}(L; P; S^V),$$

with the obvious $G$-maps

$$\sigma: S^W - V \wedge t(L; P)(V) \to t(L; P)(W)$$

as prespectrum structure maps. Here $S^V$ is the one-point compactification of $V$ and $W - V$ is the orthogonal complement of $V$ in $W$. We also define a $G$-spectrum $T(L; P)$ associated with $t(L; P)$, i.e. a $G$-prespectrum where the adjoints $\tilde{\sigma}$ of the structure maps are homeomorphisms. We first replace $t(L; P)$ by a thickened version $t^r(L; P)$ where the structure maps $\sigma$ are closed inclusions. It has as $V$'th space the homotopy colimit over suspensions of the structure maps

$$t^r(L; P)(V) = \text{holim}_{Z \subset V} \Sigma^{V-Z} t(L; P)(Z)$$

and as structure maps the compositions $(t=t(L; P))$

$$\Sigma^W - V \text{holim}_{Z \subset V} \Sigma^{V-Z} t(Z) \cong \text{holim}_{Z \subset V} \Sigma^{W-Z} t(Z) \to \text{holim}_{Z \subset W} \Sigma^{W-Z} t(Z).$$

Here the last map is induced by the inclusion of a subcategory and as such is a closed cofibration, in particular it is a closed inclusion. Furthermore since $V$ is terminal among $Z \subset V$ there is natural map $\pi: t^r(L; P) \to t(L; P)$ which is spacewise a $G$-homotopy equivalence. Next we define $T(L; P)$ by

$$T(L; P)(V) = \lim_{W \subset U} \Omega^{W-V} t^r(L; P)(W)$$

with the obvious structure maps.

We can replace $\text{THH}(L; P; S^V)$ by $\text{THH}(L; S^V)$ above and get a $G$-prespectrum $t(L)$ and a $G$-spectrum $T(L)$. These possess some extra structure which allows the definition of $\text{TC}(L)$ and we will now discuss this in some detail. For a complete account we refer to [6], see also [3].

1.3. Let $C$ be a finite subgroup of $G$ of order $r$ and let $J$ be the quotient. The $r$'th root $\rho_C: G \to J$ is an isomorphism of groups and allows us to view a $J$-space $X$ as a $G$-space $\rho_C^* X$. Recall that the free loop space $LX$ has the special property that $\rho_C LX^C \cong G LX$ for any finite subgroup of $G$. Cyclotomic spectra, as defined in [3] and [6], is a class of $G$-spectra which have the analogous property in the world of spectra. This section recalls the definition.

For a $G$-spectrum $T$ there are two $J$-spectra $T^C$ and $\Phi^C T$ each of which could be called the $C$-fixed spectrum of $T$. If $V \subset U^C$ is a $C$-trivial representation, then

$$T^C(V) = T(V)^C, \quad \Phi^C T(V) = \lim_{W \subset U} \Omega^{W-V} T(W)^C$$
and the structure maps are evident. There is a natural map \( r_C: T^C \to \Phi^C T \) of \( J \)-spectra; \( r_C(V) \) is the composition
\[
T^C(V) \simeq \lim_{W \subset U} F(S^{W-V}, T(W))^C \overset{\iota^*}{\longrightarrow} \lim_{W \subset U} F(S^{W^C-V}, T(W)^C) = \Phi^C T(V)
\]
where the map \( \iota^* \) is induced by the inclusion of \( C \)-fixed points. The difference between \( T^C \) and \( \Phi^C T \) is well illustrated by the following example.

**Example.** Consider the case of a suspension \( G \)-spectrum \( T = \Sigma^\infty G X \),
\[
T(V) = \lim_{W \subset U} \Omega^{W-V} (S^W \wedge X).
\]
We let \( E_G H \) denote a universal \( H \)-free \( G \)-space, that is \( E_G H^K \simeq * \) when \( H \cap K = 1 \) and \( E_G H^K = \emptyset \) when \( H \cap K \neq 1 \). Then on the one hand we have the tom Dieck splitting
\[
(\Sigma^\infty G X)^C \simeq \bigvee_{H \leq C} \Sigma^\infty J(\Sigma G / H (C/H)_+ \wedge_C X^H),
\]
and on the other hand the lemma shows that \( \Phi^C (\Sigma^\infty G X) \simeq \bigvee_{J} \Sigma^\infty X^C \). Moreover the natural map \( r_C: (\Sigma^\infty G X)^C \to \Phi^C (\Sigma^\infty G X) \) is the projection onto the summand \( H = C \). \( \square \)

A \( J \)-spectrum \( D \) defines a \( G \)-spectrum \( \rho_C^* D \). However this \( G \)-spectrum is indexed on the \( G \)-universe \( \rho_C^* U^C \) rather than on \( U \). To get a \( G \)-spectrum indexed on \( U \) we must choose an isometric isomorphism \( f_C: U \to \rho_C^* U^C \), then \( (\rho_C^* D)(f_C(V)) \) is the \( V \)th space of the required \( G \)-spectrum, which we denote it \( \rho_C^* D \).

We want the \( f_C \)'s to be compatible for any pair of finite subgroups, that is the following diagram should commute
\[
\begin{array}{ccc}
U & \xrightarrow{f_C} & \rho_C^* U^C \\
\downarrow f_C & & \downarrow \rho_C^* U^C \\
\rho_C^* U^C & \xrightarrow{\rho_C^* (f_C)^C} & \rho_C^* (\rho_C^* U^C)^C \\
\end{array}
\]
Moreover the restriction of \( f_C \) to the \( G \)-trivial universe \( U^G \) induces an automorphism of \( U^G \) which we request be the identity. We fix our universe,
\[
U = \bigoplus_{n \in \mathbb{Z}, n \in \mathbb{N}} \mathbb{C}(n)_\alpha,
\]
where \( \mathbb{C}(n) = \mathbb{C} \) but with \( G \) acting through the \( n \)th power map. The index \( \alpha \) is a dummy. Since \( \rho_C^* \mathbb{C}(n) = \mathbb{C}(nr) \), where \( r \) is the order of \( C \), we obtain the required maps \( f_C \) by identifying \( \mathbb{Z} = r\mathbb{Z} \).

**Definition.** ([6]) A cyclotomic spectrum is a \( G \)-spectrum indexed on \( U \) together with a \( G \)-equivalence
\[
\varphi_C: \rho_C^* \Phi^C T \to T
\]
for every finite \( C \subset G \), such that for any pair of finite subgroups the diagram
\[
\begin{array}{ccc}
\rho_C^* \Phi^C_r \rho_C^* \Phi^C_s T & \xrightarrow{\varphi_C^r \varphi_C^s} & \rho_C^* \Phi^C T \\
\downarrow \rho_C^* \Phi^C_r \varphi_C^s & & \downarrow \varphi_C^r \\
\rho_C^* \Phi^C_T & \xrightarrow{\varphi_C} & T
\end{array}
\]
commutes.
We prove in [6] that the topological Hochschild spectrum $T(L)$ defined above is a cyclotomic spectrum. The rest of this section recalls the definition of the $\varphi$-maps for $T(L)$. The definition goes back to [2] and begins with the concept of edgewise subdivision.

The realization of a cyclic space becomes a $G$-space upon identifying $G$ with $\mathbb{R}/\mathbb{Z}$, and hence $C$ may be identified with $r^{-1}\mathbb{Z}/\mathbb{Z}$. Edgewise subdivision associates to a cyclic space $Z$, a simplicial $C$-space $\text{sd}_C Z$. It has $k$-simplices $\text{sd}_C Z_k = Z_{r(k+1)-1}$ and the generator $r^{-1} + \mathbb{Z}$ of $C$ acts as $r^{k+1}$. Moreover, there is a natural homeomorphism

$$D: |\text{sd}_C Z_1| \to |Z_1|,$$

an $\mathbb{R}/r\mathbb{Z}$-action on $|\text{sd}_C Z_1|$ which extends the simplicial $C$-action, and $D$ is $G$-equivariant when $\mathbb{R}/r\mathbb{Z}$ is identified with $\mathbb{R}/\mathbb{Z}$ through division by $r$.

We now consider the case of $\text{THH}(L; X)_k$. Let us write $G_k(i_0, \ldots, i_k)$ for the pointed mapping space $F(S^{|i_0|} \wedge \ldots \wedge S^{|i_k|}, L(S^{|i_0|}) \wedge \ldots \wedge L(S^{|i_k|}) \wedge X)$. Then the $k$-simplices of the edgewise subdivision is the homotopy colimit

$$\text{sd}_C \text{THH}(L; X)_k = \text{holim}_i G_{r(k+1)-1}.$$

The $C$-action on $\text{sd}_C \text{THH}(L; X)_k$ is not induced by one on $G_{r(k+1)-1}$. We consider instead the composite functor $G_{r(k+1)-1} \circ \Delta_r$ where $\Delta_r: I^{k+1} \to (I^{k+1})^r$ is the diagonal functor. It has $C$-action and the canonical map of homotopy colimits

$$b_k: \text{holim}_i G_{r(k+1)-1} \circ \Delta_r \to \text{holim}_i G_{r(k+1)-1}$$

is a $C$-equivariant inclusion and induces a homeomorphism of $C$-fixed sets. Let $Y$ and $Z$ be two $C$-spaces and consider the mapping space $F(Y, Z)$. It is a $C$-space by conjugation and we have a natural map

$$\iota^*: F(Y, Z)^C \to F(Y^C, Z^C),$$

which takes a $C$-equivariant map $\psi: Y \to Z$ to the induced map of $C$-fixed sets. In the case at hand $\iota^*$ gives us a natural transformation

$$(G_{r(k+1)-1} \circ \Delta_r)^C \to G_k,$$

and the induced map on homotopy colimits defines a map of simplicial spaces

$$\tilde{\phi}_C.: \text{sd}_C \text{THH}(L; X)^C \to \text{THH}(L; X^C)_k.$$

We define a $G$-equivariant map

$$\phi_C(V): \rho_C^* \iota(L(V))^C \to \iota(L(f_C^{-1}(\rho_C^* V^C)))$$

as the composite

$$\rho_C^*| \text{THH}(L; S^V)^C| \overset{D}{\longrightarrow} |\text{sd}_C \text{THH}(L; S^V)^C| \overset{\tilde{\phi}_C}{\longrightarrow} |\text{THH}(L; S^\rho_C V^C)| \overset{(f_C^{-1})^*}{\longrightarrow} |\text{THH}(L; S^\rho_C V^C)|.$$

Indeed it is $G$-equivariant by [2] lemma 1.11. Next we define a $G$-map

$$\varphi_C(V): \rho_C^* T(L(V))^C \to T(f_C^{-1}(\rho_C^* V^C))$$

as the map on colimits over $W \subset U$ induced by the composition

$$\rho_C^*(\Omega^{W-V} t^*(L(W)))^C \overset{\phi_C(W)}{\longrightarrow} \rho_C^*(\Omega^{\rho_C(W-C-V)} t^*(L(W))^C \overset{f_C^{-1}}{\longrightarrow} \Omega^{f_C^{-1}(\rho_C(W-V))^C} t^*(L(f_C^{-1}(\rho_C^* W^C)))$$

Then the required maps $\varphi_C: \rho_C^* \Phi^C T \to T$ of $G$-spectra are evident in view of the definitions. Furthermore [2] 1.12 shows that the diagram which relates the $\varphi$-maps for a pair of finite subgroups of $G$ commutes. We refer to [6] for the proof that the $\varphi$-maps are $G$-equivalences.
1.4. Let $j: U^G \to U^C$ be the inclusion of the trivial $G$-universe and let $D$ be a $J$-spectrum. The underlying non-equivariant spectrum of $D$ is the spectrum $j^*D$ with its $J$-action forgotten. By abuse of notation we usually denote this $D$ again.

Let $T$ be a cyclotomic spectrum, then $r_{C^r}$ and $\varphi_{C^r}$ induce a map of $G$-spectra

$$p_{C^r}^# T_{C^r}^{C} = p_{C^r}^# (p_{C^r}^# T_{C^r}^{C}) \to p_{C^r}^# (p_{C^r}^# \Phi_{C^r} T) \to p_{C^r}^# T_{C^r}^r.$$ 

It gives a map $\Phi_{p}: T_{C^r}^{C} \to T_{C^r}^r$ of underlying non-equivariant spectra and the compatibility condition in definition 1.3 implies that $\Phi_{p} = \Phi_{rs}$. The inclusion of the fixed set of a bigger group in that of a smaller also defines a map of non-equivariant spectra $D_{p}: T_{C^r}^{C} \to T_{C^r}^C$, and these satisfies that $D_{p}D_{s} = D_{rs}$. Moreover $D_{p}\Phi_{s} = \Phi_{s}D_{p}$.

Topological cyclic homology of an FSP was defined in [2]; the presentation here is due to T. Goodwillie [5]. Let $\mathcal{I}$ be the category with $\text{ob } \mathcal{I} = \{1, 2, 3, \ldots \}$ and two morphisms $\Phi_{p}, D_{p}: n \to m$, whenever $n = rm$, subject to the relations

$$\Phi_{1} = D_{1} = \text{id}_{n}, \quad \Phi_{p} \Phi_{s} = \Phi_{rs}, \quad D_{s}D_{s} = D_{rs}, \quad \Phi_{r}D_{s} = D_{r}\Phi_{p}.$$ 

For a prime $p$ we let $I_{p}$ denote the full subcategory with $\text{ob } I_{p} = \{1, p, p^2, \ldots \}$. The discussion above shows that a cyclotomic spectrum $T$ defines a functor from $\mathcal{I}$ to the category of non-equivariant spectra, which takes $n$ to $T_{C^n}$.

**Definition.** ([2]) $TC(T) = \text{holim}_{l_{p}} T_{C^n}^{C}, \quad TC(T; p) = \text{holim}_{l_{p}^{p}} T_{C^n}^{p^*}$.

If $L$ is a functor with smash product then $TC(L)$ and $TC(L; p)$ are the connective covers of $TC(T(L))$ and $TC(T(L); p)$ respectively. It is often useful to have the definition of $TC(T; p)$ in the form it is given in [2],

$$TC(T; p) \cong \text{holim}_{D_{p}} [\text{holim}_{l_{p}^{p}} T_{C^n}^{p^*}]^{h(\Phi_{p})} \cong [\text{holim}_{l_{p}^{p}} T_{C^n}^{p^*}]^{h(D_{p})}.$$ 

Here $(D_{p})$ is the free monoid on $D_{p}$ and $X^{h(D_{p})}$ stands for the $(D_{p})$-homotopy fixed points of $X$. It is naturally equivalent to the homotopy fiber of $1 - D_{p}$.

The functor $TC(-)$ is really not a stronger invariant than the $\text{TC}(-; p)$'s. Indeed we have the following result, which will be proved elsewhere.

**Proposition.** The projections $TC(T) \to TC(T; p)$ induce an equivalence of $TC(T)$ with the fiber product of the $TC(T; p)$'s over $T$. Moreover the $p$-complete theories agree, $TC(T)_{p} \simeq TC(T; p)_{p}^{\wedge}$. □

**Remark.** In [2] the authors define a space $TC(L; p)$ and a $\Gamma$-space structure on it. Furthermore they show that the cyclotomic trace $tr_{k}(L) \to TC(L; p)$ is a map of $\Gamma$-spaces. We show in [6] that the spectrum $TC(L; p)$ defined above is equivalent to the one determined by the $\Gamma$-space structure. □

1.5. Stable $K$-theory of simplicial rings was defined by Waldhausen in [15], see also [8]. We conclude this paragraph with the definition of stable $TC$ of a FSP and leave it to reader to see that stable $K$-theory also may be defined in this generality.

**Definition.** Let $P$ be an $L$-bimodule and $K$ a space. The shift $P[K]$ of $P$ by $K$ is the functor given by $P[K](X) = K \wedge P(X)$ with structure maps

$$i_{X,Y}^{P[K]} = \text{id}_{K} \wedge i_{X,Y}^{P} \circ \text{tw} \wedge \text{id}_{P(Y)}, \quad r_{X,Y}^{P[K]} = \text{id}_{K} \wedge r_{X,Y}^{P}.$$ 

We shall write $P[n]$ for $P[S^{n}]$.

We define a new FSP denoted $L \oplus P$ which is to be thought of as an extension of $L$ by a square zero ideal $P$. 5
**Definition.** Let \( L \) be an FSP and \( P \) an \( L \)-bimodule. We define the extension of \( L \) by \( P \) as \( L \oplus P(X) = L(X) \vee P(X) \) with multiplication

\[
L \oplus P(X) \cong L \oplus P(y) \cong L(X) \vee L(X) \vee P(X) \vee P(X) \vee P(X) \vee P(Y) \\
\rightarrow L(X \vee Y) \vee P(X \vee Y) \vee P(X \vee Y) \rightarrow L \oplus P(X \vee Y).
\]

The first map is the canonical homeomorphism, the second is \( \mu_{X,Y} \vee \tau_{X,Y} \vee \ast \) and the last is convolution. Finally the unit in \( L \oplus P \) is the composite

\[
X \rightarrow L(X) \rightarrow L \oplus P(X).
\]

One verifies immediately that \( L \oplus P \) is in fact an FSP and that it contains \( L \) as a retract. We shall write \( \widetilde{TC}(L \oplus P) \) for the homotopy fiber of the induced retraction \( TC(L \oplus P) \rightarrow TC(L) \).

**Lemma.** If \( K \) is contractible then so is \( \widetilde{TC}(L \oplus P[K]) \). Furthermore a contraction of \( K \) induces one of \( \widetilde{TC}(L \oplus P[K]) \).

**Proof.** Let us write \( F \) instead of \( L \oplus P[K] \). If \( h: I_+ \wedge K \rightarrow K \) is a contraction we can define \( h(X): I_+ \wedge F(X) \rightarrow F(X) \) by the composition

\[
I_+ \wedge (L(X) \vee K \wedge P(X)) \cong I_+ \wedge L(X) \vee I_+ \wedge K \wedge P(X) \xrightarrow{pr_2 \wedge h \wedge id} L(X) \vee K \wedge P(X).
\]

It is compatible with the multiplication and unit in \( F \), that is the following diagrams commute

\[
\begin{array}{ccc}
I_+ \wedge (F(X) \wedge F(Y)) & \xrightarrow{id \wedge \mu_{X,Y}} & I_+ \wedge F(X \wedge Y) \\
\downarrow \Delta \wedge id & & \downarrow h_{X,Y} \\
(I \times I)_+ \wedge F(X) \wedge F(Y) & & F(X \wedge Y) \\
\downarrow \operatorname{id} \wedge \operatorname{tw} \operatorname{id} & & \uparrow \mu_{X,Y} \\
I_+ \wedge F(X) \wedge I_+ \wedge F(Y) & \xrightarrow{h_{X,Y} \wedge h_{X,Y}} & F(X \wedge F(Y).)
\end{array}
\]

and

\[
\begin{array}{ccc}
I_+ \wedge X & \xrightarrow{id \wedge 1X} & I_+ \wedge F(X) \\
\downarrow \operatorname{pr}_2 & & \downarrow h(X) \\
X & \xrightarrow{1X} & F(X).
\end{array}
\]

Therefore the composition

\[
I_+ \wedge (F(S^0) \wedge \ldots \wedge F(S^k)) \xrightarrow{\operatorname{tw}(\Delta \wedge id)} I_+ \wedge F(S^0) \wedge \ldots \wedge I_+ \wedge F(S^k) \\
\xrightarrow{h(S^0) \wedge \ldots \wedge h(S^k)} F(S^0) \wedge \ldots \wedge F(S^k)
\]

gives rise to a cyclic map \( h_V: I_+ \wedge \operatorname{THH}(F; F; S^V)_* \rightarrow \operatorname{THH}(F; F; S^V)_* \), whose realization is a \( G \)-equivariant homotopy

\[
h_V: I_+ \wedge t(F)(V) \rightarrow t(F)(V).
\]

Furthermore these are compatible with the structure maps in the prospectrum such that we get a \( G \)-equivariant homotopy

\[
H: I_+ \wedge T(F) \rightarrow T(F).
\]

This gives in turn a homotopy \( I_+ \wedge TC(F) \rightarrow TC(F) \) from the identity to the retraction onto the image of \( TC(L) \). \( \square \)
If we apply $\tilde{\text{TC}}(L \oplus P[n])$ to the cocartesian square of spaces

$$
\begin{array}{ccc}
S^n & \longrightarrow & D^{n+1}_+ \\
\downarrow & & \downarrow \\
D^{n+1}_- & \longrightarrow & S^{n+1}.
\end{array}
$$

we get a map from $\tilde{\text{TC}}(L \oplus P[n])$ to the homotopy limit of

$$
\tilde{\text{TC}}(L \oplus P[D^{n+1}]) \to \tilde{\text{TC}}(L \oplus P[S^{n+1}], p) \leftarrow \tilde{\text{TC}}(L \oplus P[D^{n+1}]).
$$

By the lemma the radial contractions of the discs $D^{n+1}_+$ give a preferred contraction of $\tilde{\text{TC}}(L \oplus P[D^{n+1}])$. Hence we obtain a natural map from the homotopy limit above to $\Omega \tilde{\text{TC}}(L \oplus P[n + 1])$. All in all we get a stabilization map

$$
\sigma : \tilde{\text{TC}}(L \oplus P[n]) \to \Omega \tilde{\text{TC}}(L \oplus P[n + 1])
$$

which is natural in $L$ and $P$.

**Definition.** Let $L$ be an FSP and $P$ an $L$-bimodule. Then

$$
\text{TC}^S(L; P) = \text{holim}_{n} \Omega^{n+1} \tilde{\text{TC}}(L \oplus P[n]),
$$

with the colimit taken over the stabilization maps.

### 2. Stable Approximation of $\text{TC}(L \oplus P)$

2.1. In the rest of this paper the prime $p$ is fixed and we shall always consider the functor $\text{TC}(-; p)$ rather than $\text{TC}(-)$.

Recall that by definition $L \oplus P(S^i) = L(S^i) \vee P(S^i)$. Thus we can decompose the smash product

$$
L \oplus P(S^0) \wedge \ldots \wedge L \oplus P(S^n)
$$

into a wedge of summands of the form

$$
F_0(S^0) \wedge \ldots \wedge F_k(S^k),
$$

where $F_i = L, P$. A summand where $\# \{i | F_i = P\} = a$ will be called an $a$-configuration and the one-point space $\ast$ will be considered an $a$-configuration for any $a \geq 0$.

Recall from 1.3 the functor $G_k = G_k(L \oplus P; X)$ whose homotopy colimit is $\text{THH}(L \oplus P; X)_k$. The $a$-configurations define subspaces

$$
G_{a,k}(i_0, \ldots, i_k) \subset G_k(i_0, \ldots, i_k)
$$

preserved under $G_k(f_0, \ldots, f_k)$, i.e. we get a functor $G_{a,k} = G_{a,k}(L \oplus P; X)$. The spaces

$$
\text{THH}_a(L \oplus P; X)_k = \text{holim}_{i_{k+1}} G_{a,k}(L \oplus P; X)
$$

form a cyclic subspace $\text{THH}_a(L \oplus P; X) \subset \text{THH}(L \oplus P; X)$, with realization $\text{THH}_a(L \oplus P; X)$. Like in 1.2 we can define a $G$-prespectrum $\Sigma(L \oplus P)$ and a $G$-spectrum $T_a(L \oplus P)$. Then $T_a(L \oplus P)$ is a retract of $T(L \oplus P)$. We show below that as a $G$-spectrum $T(L \oplus P)$ is the wedge sum of the $T_a(L \oplus P)$'s.
Lemma. Let $j$ be a $G$-prespectrum and let $J$ be the $G$-spectrum associated with $j^\tau$. If $J^\Gamma \simeq *$ for any finite subgroup $\Gamma \subset G$ and $j(V)^G \simeq *$ for any $V \subset U$ then $J \simeq_G *$.

Proof. Let $F$ be the family of finite subgroups of the circle, then $J$ is $F$-contractible. Since $J \wedge EF_+ \to J$ is an $F$-equivalence, $J \wedge EF_+$ is also $F$-contractible. However $J \wedge EF_+$ is $G$-equivalent to an $F$-CW-spectrum and therefore it is in fact $G$-contractible by the $F$-Whitehead theorem, [9] p.63. Now

$$(J \wedge EF_+(V) \cong \lim_W \Omega^W j^\tau(V + W) \wedge EF_+),$$

and $j^\tau(V) \wedge EF_+ \to j^\tau(V)$ is an $G$-equivalence since $j(V)^G \simeq *$. Therefore $J \simeq_G J \wedge EF_+$ and we have already seen that the latter is $G$-contractible. □

Lemma. Let $H$ be a compact Lie group, let $X$ a finite $H$-CW-complex and let $Y_a$ a family of $H$-spaces. For $K \leq H$ a closed subgroup we let $n(K) = \min_a \{\text{conn}(Y^K_a)\}$. Then the inclusion

$$\bigvee a F(X,Y_a)^H \to F(X,\bigvee a Y_a)^H$$

is $2 \min\{n(K) - \dim(X^K)|K \leq H\} + 1$-connected.

Proof. The inclusion above fits into a commutative square

$$\begin{array}{ccc}
\bigvee a F(X,Y_a)^C & \longrightarrow & F(X,\bigvee a Y_a)^C \\
\downarrow & & \downarrow \\
\prod a F(X,Y_a)^C & \xrightarrow{\simeq} & F(X,\prod a Y_a)^C,
\end{array}$$

where $\prod'$ is the weak product, i.e. the subspace of the product where all but a finite number of coordinates are at the basepoint. The lower horizontal map is a homeomorphism because $X$ is finite, and the connectivity of the vertical maps may be estimated by elementary equivariant obstruction theory. For example the connectivity of an equivariant mapping space satisfies

$$\text{conn}(F(X,Y)^H) \geq \min\{\text{conn}(Y^K) - \dim(X^K)|K \leq H\}.$$ 

Therefore the left vertical map is $2 \min\{n(K) - \dim(X^K)|K \leq H\} + 1$-connected. □

Proposition. $T(L \oplus P) \simeq_G \bigvee a T_a(L \oplus P)$.

Proof. We apply the first lemma with $j$ the $G$-prespectrum whose $V$th space is the homotopy fiber of the inclusion

$$\bigvee a t_a(L \oplus P)(V) \to t(L \oplus P)(V).$$

We first consider a finite subgroup $\Gamma \subset G$ and show that $J^\Gamma \simeq *$. It suffices to show that $j(V)^G$ is $\dim(V^\Gamma) + k(V,C)$-connected, where $k(V,C) \to \infty$ as $V$ runs through the f.d. sub inner product spaces of $U$, for any subgroup $C \subset \Gamma$. We use edgewise subdivision to get a simplicial $C$-action, that is we can identify $j(V)^C$ with the homotopy fiber of

$$|\bigvee a \text{sd}_C THH_a(L \oplus P; S^V)^C| \to |\text{sd}_C THH(L \oplus P; S^V)^C|.$$ 

As in the 1.3 we consider the diagonal functor $\Delta_r: I^{k+1} \to (I^{k+1})^r$. Then the second lemma shows that the inclusion

$$\bigvee a (G_{a,r(k+1)-1} \circ \Delta_r(i_0, \ldots, i_k))^C \to (G_{r(k+1)-1} \circ \Delta_r(i_0, \ldots, i_k))^C$$
is $2 \dim(V^C) - 1$-connected. By [1] theorem 1.5 the same is true for the homotopy colimits over $I^{k+1}$. Hence the inclusion map

$$\bigvee_{a} \text{sd}_{C} \text{THH}_{a}(L \oplus P; S^{V^C})_{k} \to \text{sd}_{C} \text{THH}(L \oplus P; S^{V^C})_{k}$$

is $2 \dim(V^C) - 1$-connected. Finally the spectral sequence of [13] shows that the induced map on realizations is $2 \dim(V^C) - 1$-connected. It follows that $J^1 \simeq \ast$.

We have only left to show that $j(V)^{G} \simeq \ast$. If $X_{*}$ is a cyclic space, then $|X_{*}|^{G}$ is homeomorphic to the subspace $\{ x \in X_{0} | s_{0} x = \tau_{1} s_{1} x \}$ of the $0$-simplices. For the domain and the codomain of $j(V)$ this is $S^{V^G}$ and $j(V)$ is the identity. 

\[ \square \]

2.2. Let us write $a = p^sk$ with $(k, p) = 1$ and denote $T_{a}(L \oplus P)$ by $T_{a}^{k}(L \oplus P)$. Then the cyclotomic structure map $\varphi = \varphi_{C_p}$ induces a $G$-equivalence

$$\varphi_{*} : \rho_{C_p}^{k} \Phi^{C_p} T_{a}^{k}(L \oplus P) \to T_{a-1}^{k}(L \oplus P), \ s \geq 0,$$

where for convenience $T_{a}^{k} (L \oplus P)$ denotes the trivial $G$-spectrum $\ast$.

**Lemma.** i) The cyclotomic structure map induces a map of underlying non-equivariant spectra

$$T_{a}^{k}(L \oplus P[n])^{C_p} \to T_{a}^{k}(L \oplus P[n])^{C_p-\ast}$$

which is $kpn$-connected.

ii) $T_{a}^{k}(L \oplus P[n])^{C_p}$ is $kn$-connected.

**Proof.** Let $\tilde{EG}$ be the mapping cone of the map $\pi : EG_{+} \to S^{0}$ which collapses $EG$ to the non-basepoint of $S^{0}$. It comes with a $G$-map $\iota : S^{0} \to \tilde{EG}$ and a $G$-null homotopy of the composition

$$EG_{+} \xrightarrow{\simeq} S^{0} \xrightarrow{\iota} \tilde{EG}. $$

We can also describe $\tilde{EG}$ as the unreduced suspension of $EG$ and $\iota$ as the inclusion of $S^{0}$ as the two cone vertices. Finally we note that $\tilde{EG}$ is non-equivariantly contractible while $\tilde{EG}^{C} = S^{0}$ for any non-trivial subgroup $C \leq G$.

Let us write $T_{a}$ for $T_{a}^{k}(L \oplus P[n])$. We can smash the sequence above with $T_{a}$ and take $C_p$-fixed points. Then we get maps of underlying non-equivariant spectra

$$[EG_{+} \wedge T_{a}]^{C_p} \xrightarrow{\pi_{*}} T_{a}^{C_p} \xrightarrow{\iota_{*}} [\tilde{EG} \wedge T_{a}]^{C_p}$$

and a preferred null homotopy of their composition. These data specifies a map from $[EG_{+} \wedge T_{a}]^{C_p}$ to the homotopy fiber of $\iota_{*}$ and this an equivalence.

We identify the right hand term. Recall the natural map $r_{C_p} : T_{a}^{C_p} \to \Phi^{C_p} T_{a}$ from 1.3. It factors as a composition

$$T_{a}^{C_p} \xrightarrow{\pi_{*}} [\tilde{EG} \wedge T_{a}]^{C_p} \xrightarrow{\iota_{C_p}} \Phi^{C_p} T_{a},$$

where $r_{C}(V)$ is induced from the restriction map

$$F(S^{W-V}, \tilde{EG} \wedge T_{a}(W))^{C_p} \to F(S^{W^{C_p}-V}, T(W)^{C_p}).$$

Moreover $r_{C_p}(V)$ is a fibration with fiber the equivariant (pointed) mapping space

$$F(S^{W-V}/S^{W^{C_p}-V}, \tilde{EG} \wedge T(W))^{C_p}.$$
If we regard $W$ as a $C_p$-space, then $W^{C_p}$ is the singular set, so $S^{W−V}/S^{W^{C_p}−V}$ is a free $C_p$-CW-complex in the pointed sense. Since $EG$ is non-equivariantly contractible it follows that $rC_p$ is a $C_p/C_p$ equivalence. The map $\Phi\rho$ of underlying non-equivariant spectra defined in 1.4 restricts to a map

$$T_s^{C_p^{fr/C_p}} \xrightarrow{C_p^{fr/C_p}} (\Phi^{C_p T_s})^{C_p^{fr/C_p}} = (\rho^{\# C_p T_s})^{C_p^{fr-1}} \xrightarrow{C_p^{fr-1}} T_s^{C_p^{fr-1}}.$$  

Our calculation above shows that its homotopy fiber is equivalent to the underlying non-equivariant spectrum of $[EG_+ \wedge T_s]^{C_p^{fr}}$. We contend that this is as highly connected as is $T_s$. Indeed the skeleton filtration of $EG$ gives rise to a first quadrant spectral sequence

$$E^2_{s,t} = H_s(C_p^{fr}; \pi_t(T_s)) \Rightarrow \pi_{s+t}([EG_+ \wedge T_s]^{C_p^{fr}}),$$  

where $\pi_t(T_s)$ is a trivial $C_p^{fr}$-module. The identification of the $E^2$-term uses the transfer equivalence of [9] p. 89. □

**Proposition.** In the stable range $\leq 2n$ we have

$$\tilde{TC}(L \oplus P[n]) \simeq_{2n} \varinjlim \pi_1(T(L \oplus P[n]; p)^{C_p^{fr}},$$  

with the limit taken over the inclusion maps $D$.

**Proof.** We get from the connectivity statements in the lemma that

$$\tilde{T}(L \oplus P[n])^{C_p^{fr}} \simeq_{2n} T_1(L \oplus P[n])^{C_p^{fr}} = \bigvee_{s=0}^{\infty} T_s^{C_p^{fr}} = \bigvee_{s=0}^{r} T_s^{C_p^{fr}}.$$

Under these equivalences $\Phi: \tilde{T}(L \oplus P[n])^{C_p^{fr}} \to \tilde{T}(L \oplus P[n])^{C_p^{fr-1}}$ becomes projection onto the first $r$ summands. Therefore

$$\tilde{TC}(L \oplus P[n]; p) = \{\varinjlim \tilde{T}(L \oplus P[n])^{C_p^{fr}}\}^{h(D)} \simeq_{2n} \prod_{t=0}^{\infty} T_0^{C_p^{fr}}.$$  

The latter spectrum is naturally equivalent to the homotopy limit stated above. □

**Remark.** When $P = L$ there is an unstable formula for $\tilde{TC}(L \oplus L[n])$. It was found in [6] and used to evaluate TC of rings of dual numbers over finite fields.

3. Free cyclic objects

3.1. In this paragraph we examine the cyclic spaces $t_1(L \oplus P)(V)$, we introduced in 2.2. They turn out to be the free cyclic spaces generated by the simplicial spaces $t(L; P)(V)$, from 1.2. First we study free cyclic objects.

Suppose $K: I \to J$ is a functor between small categories and $\mathbb{C}$ a category which have all colimits. Then the functor $K^*: \mathbb{C}^J \to \mathbb{C}^I$ has a left adjoint $F$. If $X: I \to \mathbb{C}$ is a functor then

$$FX(j) = \lim_{\longrightarrow} ((K \downarrow j) \xrightarrow{pr_1} I \xrightarrow{X} \mathbb{C}),$$

where $(K \downarrow j)$ is the category of objects $K$-over $j$. It is called the left Kan extension of $X$ along $K$, cf. [10]. As an instance of this construction suppose $I$ and $J$ are monoids, i.e. categories with one object, and $\mathbb{C}$ the category of (unbased) spaces. Then a functor $X: I \to \mathbb{C}$ is just an $I$-space and $FX$ is the $J$-space $J \times_1 X$. 10
Definition. Let $X_\bullet$ be a simplicial object in $\mathcal{C}$. The free cyclic object generated by $X_\bullet$ is the left Kan extension of $X_\bullet$ along the forgetful functor $K: \Delta^{op} \to \mathbf{A}$. It is denoted $FX_\bullet$.

If $X$ is an object in $\mathcal{C}$ and $S$ is a set, then we let $S \times X$ denote the coproduct of copies of $X$ indexed by $S$. We give a concrete description of $FX_\bullet$.

Lemma. Let $C_{n+1} = \{1, \tau_n, \tau_n^2, \ldots, \tau_n^n\}$. Then $FX_\bullet$ has $n$-simplices

$$FX_n \cong C_{n+1} \times X_n,$$

and the cyclic structure maps are

$$d_i(\tau_n^k x) = \tau_n^k d_{i+s} x \quad \text{if } i + s \leq n$$

$$s_i(\tau_n^k x) = \tau_n^{k+1}s_{i+s} x \quad \text{if } i + s \leq n$$

$$t_n(\tau_n^k x) = \tau_n^{k-1}x.$$

All indices are to be understood as the principal representatives modulo $n+1$.

Proof. Both $\Delta$ and $\mathbf{A}$ has objects the finite ordered sets $n = \{0, \ldots, n\}$ but $\mathbf{A}$ has more morphism than $\Delta$. Specifically $\Lambda(n, m) = \Delta(n, m) \times \text{Aut}_\Delta(n)$ and $\text{Aut}_\Delta(n)$ is a cyclic group of order $n+1$. As a generator for $\text{Aut}_\Delta(n)$ we choose the cyclic permutation $\tau_n: n \to n; \tau_n(i) = i - 1 (\text{mod } n+1)$.

Consider the full subcategory $C(n) \subset (K \downarrow n)$ whose objects are the automorphisms $K \nrightarrow n$, i.e. $\text{ob } C(n) = C_{n+1}$. The restriction of colimits comes with a map

$$\lim(C(n) \xrightarrow{pr} \Delta^{op} \xrightarrow{X_n} \mathbb{C}) \to \lim((K \downarrow n) \xrightarrow{pr} \Delta^{op} \xrightarrow{X_n} \mathbb{C}) = FX_n,$$

and from the definitions one may readily show that this is an isomorphism. Since in $\Delta^{op}$ there are no automorphisms of $n$ apart from the identity, the category $C(n)$ is a discrete category, i.e. any morphism is an identity. We conclude that

$$FX_n \cong \prod_{\text{ob } C(n)} X_n = C_{n+1} \times X_n.$$

It is straightforward to check that the cyclic structure maps are as claimed. \qed

Example. Suppose $\mathbb{C}$ is the category of commutative rings, where the coproduct is tensor product of rings, and $R = \{0\}$ is a constant simplicial ring. Then the complex associated with $FR$ is the standard Hochschild complex $Z(\mathbb{C})$ whose homology is $H_1(R)$.

3.2. We now take $\mathbb{C}$ to be the category of pointed topological spaces and study the relation between $F$ and realization.

Lemma. There is a natural $G$-homeomorphism $|FX_\bullet| \cong G_+ \wedge |X_\bullet|$.

Proof. Consider the standard cyclic sets $\Lambda[n] = \Delta(-, n)$ and their realizations $\Lambda^n$. From [7], 3.4 we know that as cocyclic spaces $\Lambda^* \cong G \times \Delta^*$, so we may view $\Lambda^*$ as a cocyclic $G$-space. Now suppose $Y$ is a (based) $G$-space. We can define a cyclic space $C_\ast(Y)$ as the equivariant mapping space

$$C_\ast(Y) = F_G(\Lambda^*, Y),$$

with the compact open topology. Then one immediately verifies that $C_\ast$ is right adjoint to the realization functor $|-|$. The realization functor for simplicial spaces also has a right adjoint. It is given as $S_\ast(X) = F(\Delta^*, X)$ with the compact open topology. Finally the forgetful functor $U$ from $G$-spaces to spaces is right adjoint to the functor $G_+ \wedge -$.

By a very general principle in category theory called conjunction, to prove the lemma we may as well show that $S_\ast(UY) = K^*C_\ast(Y)$ for any $G$-space $Y$. But this is evident since $F_G(G_+ \wedge X, Y) \cong F(X, UY)$ \qed
Proposition. There is a natural equivalence of $G$-spectra

$$G_+ \wedge T(L; P) \simeq_G T_1(L \oplus P).$$

The $V'$th space in the smash product $G$-spectrum on the left is naturally homeomorphic to $\lim_{W} \Omega^{W-V}(G_+ \wedge t^\tau(L; P)(W))$, where $G$ acts diagonally on $G_+ \wedge t^\tau(L; P)(W)$.

Proof. The smash product $P(S^{i_0}) \wedge L(S^{i_1}) \wedge \ldots \wedge L(S^{i_k})$ is a 1-configuration, cf. 2.1. Thus we have an inclusion map $\text{THH}(L; P; X)_k \hookrightarrow \text{THH}_1(L \oplus P; X)_k$ and these commutes with the simplicial structure maps. By definition we get a map of cyclic spaces

$$j(X)_! : F \text{THH}(L; P; X) \to \text{THH}_1(L \oplus P; X),$$

and lemma 3.2 shows that on realizations this gives rise to a $G$-equivariant map

$$j(X)_! : G_+ \wedge \text{THH}(L; P; X) \to \text{THH}_1(L \oplus P; X).$$

When $X$ runs through the spheres $S^{1}$ these maps form a map $j$ of $G$-prespectra. Let us write $G_+ \wedge t^\tau(L; P)$ for the $G$-spectrum whose $V'$th space is the colimit

$$\lim_{W \subseteq U} \Omega^{W-V}(G_+ \wedge t^\tau(L; P)(W)).$$

Then $j$ induces a map $J : G_+ \wedge t^\tau(L; P) \to T_1(L \oplus P)$ and an argument completely analogous to the proof of proposition 2.1 shows that this is a $G$-equivalence. Finally the canonical inclusion

$$G_+ \wedge t^\tau(L; P)(V) \to G_+ \wedge T(L; P)(V)$$

gives a map $G_+ \wedge t^\tau(L; P) \to G_+ \wedge T(L; P)$ and this is a homeomorphism, cf. the appendix. □

3.3. Before we prove our main theorem we need the following key lemma, also used extensively in [6].

Lemma. Let $T$ be a $G$-spectrum. Then there is a natural equivalence of non-equivariant spectra

$$[T \wedge G_+]^{G_{r'}} \simeq T \vee \Sigma T,$$

and the inclusion $D : [T \wedge G_+]^{G_{r'}} \hookrightarrow [T \wedge G_+]^{G_{r'-1}}$ becomes $p \vee \text{id}$. Here $p$ denotes multiplication by $p$.

Proof. The Thom collapse $t : S^2 \to S^{2r} \wedge G_+$ of $S(C) \subset C$ gives rise to a $G$-equivariant transfer map

$$\tau : F(G_+, \Sigma T) \to G_+ \wedge T$$

which is a $G$-homotopy equivalence, cf. [9], p.89. There is a cofibration sequence of $C_{p_{r'}}$-spaces

$$C_{p_{r'}} \hookrightarrow G_+ \to C_{p_{r'}} \wedge S^1$$

where $S^1$ is $C_{p_{r'}}$-trivial. We may apply $F C_{p_{r'}}(\cdot, \Sigma T)$ and get a cofibration sequence of spectra

$$F(S^1, \Sigma T) \longrightarrow F C_{p_{r'}}(G_+, \Sigma T) \xrightarrow{ev_{\xi}} \Sigma T.$$ Finally $ev_{\xi}$ is naturally split by the adjoint of the $G$-action $G_+ \wedge \Sigma T \to \Sigma T$. □

Proof of theorem. If we compare proposition 3.2 and lemma 3.3 we find that

$$T_1(L \oplus P)^{G_{r'}} \simeq T(L; P) \vee \Sigma T(L; P).$$

Now holim of a string of maps

$$\ldots \xrightarrow{f_{n}} X_n \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_{2}} X_2 \xrightarrow{f_{1}} X_1 \xrightarrow{f_{0}} X_0$$

where every $f_i = pg_i$ for some $g_i$ vanishes after $p$-completion, so by proposition 2.2 and lemma 3.3 we get

$$\text{THH}(L \oplus P[n]) \simeq_2 \Sigma T(L; P[n]).$$

The functor $T(L; P)$ is linear in the second variable, cf. [12] 2.13, so therefore

$$\Omega^{n+1} \text{THH}(L \oplus P[n]) \simeq_2 \Omega^{n+1} \Sigma T(L; P[n]) \simeq T(L; P).$$

It remains only to check that the stabilization maps defined in 1.5 induce an equivalence of $T(L; P)$. They do. □
Appendix

A.1. Let $\mathcal{C}$ be either of the categories $\Delta$ or $\Lambda$ and let $X: \mathcal{C} \to \text{Top}_\ast$ be a functor to pointed spaces. We define a new functor $\tilde{X}: \mathcal{C} \to \text{Top}_\ast$ by the homotopy colimit

$$\text{holim}((− \downarrow \mathcal{C})^{\text{op}} \Longrightarrow \text{Top}_\ast),$$

where $(\mathfrak{n} \downarrow \mathcal{C})$ is the category under $\mathfrak{n}$, cf. [10]. If $\theta: \mathfrak{n} \to \mathfrak{m}$ is a morphism in $\Delta$ (not $\mathcal{C}$), which is surjective, then $\theta^* : (\mathfrak{m} \downarrow \mathcal{C}) \to (\mathfrak{n} \downarrow \mathcal{C})$ is an inclusion functor. In general inclusions of index categories induces closed cofibrations on homotopy colimits. In particular $\theta^*: \tilde{X}_m \to \tilde{X}_n$ is a closed cofibration, so $\tilde{X}$ is good in the sense of [14]. Moreover we have a homotopy equivalence $\tilde{X}_n \to X_n$ because id: $\mathfrak{n} \to \mathfrak{n}$ is initial in $(\mathfrak{n} \downarrow \mathcal{C})$.

A.2. This section explains a technical point in the passage from $G$-prespectra to $G$-spectra. Let $GPU$ denote the category of $G$-prespectra indexed on the universe $U$ and let $GSU$ be the full subcategory of $G$-spectra. In [9] the authors prove that the forgetful functor $l: GSU \to GPU$ has a left adjoint $L: GPU \to GSU$. We call this functor spectrification and if $t \in GPU$ then we call $Lt$ the associated $G$-spectrum. Such a functor is needed since many constructions such as $X \wedge -$ and any (homotopy) colimits do not preserve $G$-spectra. However $L$ has the serious drawback that in general it looses (weak) homotopy type, i.e. the homotopy type of $(Lt)(V)$ cannot be described in terms of that of the spaces $t(W)$. To control the homotopy type the $G$-prespectrum $t$ has to be an inclusion $G$-prespectrum, that is the structure maps $\sigma: t(V) \to \Omega^{W-V} t(W)$ must be inclusions, then

$$(Lt)(V) = \lim_{W \subset U} \Omega^{W-V} t(W).$$

This is the case for example if the adjoints $\sigma: \Sigma^{W-V} t(V) \to t(W)$ are closed inclusions. The thickening functor $(-)^{\tau}$ defined in 1.2 produces $G$-spectra of this kind. Therefore $L(t^{\tau})$ has the right homotopy type.

If $a: GPU \to GPU$ is a functor we define $A: GSU \to GSU$ as the composite functor $Lal$ and if $a$ has a right adjoint $b$, then $B$ is the right adjoint of $A$. Suppose $b$ preserves $G$-spectra, then $b(tT) \cong b(T)$ for any $T \in GSU$. By conjugation we get

$$A(Lt) \cong La(t)$$

for any $t \in GPU$. The functors $a$ we consider take a $G$-prespectrum, whose structure maps $\sigma$ are closed inclusions, to a $G$-prespectrum of the same kind. Hence the homotopy type of $La(t^{\tau})$ and therefore $A(L(t^{\tau}))$ may be calculated. This shows that all $G$-spectra considered in this paper have the right homotopy type.

References