HOMEWORK 3

due Wednesday, July 16, at 10:45am in 36-112

Reminder: You are encouraged to work with others, but each student must write up his or her own solutions. List the names of all collaborators on the front page of the problem set.

Part I

1. [5 pt] Use bisection and Newton's method to find a zero of $x^2 = \cos x$ on $[0, 1]$, correct to 8 decimal places.

2. [5 pt] Find the Taylor's expansion at 0 of the following functions:

   (a) $\frac{1}{1-2x}$

   (b) $\cos(x^2)$

   (c) $\sin(x + \pi/4)$

   (d) e^{2x^3}

   (e) $\ln(1 + x)$

Part II

3. [4 pt] Newton's method does not always work: try to use it to find a zero of $f(x) = x^{1/3}$ with initial value $x_0 = 1$.

4. [6 pt] This exercise proves that two *different* functions can have *the same* Taylor expansion at a point.

   (a) Use the l'Hôpital rule to prove that

   $$\lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

   for all $n \geq 0$.

   (b) Define the function

   $$f(x) = \begin{cases} 0 & : \quad x \leq 0 \\ e^{-1/x} & : \quad x > 0 \end{cases}$$

   (c) Prove that for $x > 0$,

   $$f^{(n)}(x) = e^{-1/x} p_n(1/x),$$

   where p_n is a polynomial.

(d) Use (a) to prove that $f(x)$ is smooth (differentiable infinitely many times), and give a different function which has the same derivatives at 0.

5. [20 pt] For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, define the *generalized binomial coefficient* as

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}.$$

(a) Check that when $n \in \mathbb{N}$, we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(b) Compute the k-th derivative of the function $(1+x)^\alpha$.

(c) Prove the *generalized binomial theorem*:

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

(d) Find the Taylor polynomial of degree 4 corresponding to $\sqrt{1+x}$.

(e) By using the identity

$$(a+b)^n = a^n(1+b/a)^n$$

and the generalized binomial theorem, prove the usual *binomial theorem*:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

(f) Prove that

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

(g) The *Pascal triangle* is an (infinite) triangle whose rows are indexed by $n = 0, 1, 2, \ldots$, whose columns (which run diagonally in the SW direction) are indexed by $k = 0, 1, 2, \ldots, n$, with $\binom{n}{k}$ in n-th row and k-th column. Use the expression for $\binom{n}{k}$ from (a) to draw the Pascal triangle with rows up to $n = 4$.

(h) Prove that

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Explain how to use this formula to find the entries of Pascal triangle recursively. Use this method to draw the Pascal triangle with rows up to $n = 7$.

(i) Expand

$$(a+b)^7.$$