Mini-Seminar: Basic Quasi-categories

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April 14,21,28 at 11am in Room 2-143 (c/o STAGE)

André Joyal has for a few years now (cf. [Joy02]) been developing a theory of quasi-categories, which are remarkably simple models of $(\infty, 1)$-categories. In other words, quasi-categories model the idea of weak ∞-categories whose k-morphisms are invertible for $k > 1$. Quasi-categories are completely combinatorial in nature. While Joyal uses model-categorical techniques to justify his (quite true) statement that "most concepts and results of category theory can be extended to quasi-categories," it is possible to prove most if not all of the key results using nothing but simple combinatorics. This mini-course aims to give proofs of the basic theorems of quasi-category theory, as well as giving some indications of how they might be used to study higher stacks. In particular, one can (I think) use quasi-categories to develop a good theory of n-stacks that would plug in to Carlos Simpson's "geometric n-stacks" framework (see [Sim96], or my BAGS talk this past February).

The basic outline of the topics I would like to cover are as follows:

- Motivation of quasi-categories for higher stacks from the ordinary 2-category of stacks.
- Definitions of quasi-categories, n-quasi-categories, and quasi-functors.
- Development of a good notion of equivalence of quasi-categories, quasi-isomorphisms in a quasi-category, and "truncation" functors from quasi-categories to n-quasi-categories.
- Discussion of limits and colimits, which for quasi-categories are naturally "homotopical" (e.g. we have 2-fibre products, not fibre products).
- Development of the theory of "fibred quasi-categories," what are referred to as "left fibrations," including defining very natural quasi-categories of left fibrations over a fixed base.
- Comparison of the quasi-category of left fibrations with the enriched structure on the corresponding full subcategory of a slice of SSets.
- A natural Yoneda lemma.
- Adjoint quasi-functors.
- Descent and stacks (if there is time; this will likely be nonrigorous if I get to it, so cf. [Lur03])
- The derived quasi-category and stable quasi-categories (again, nonrigorous and only if there is time)

The intended audience for these talks are people (like me) who are unfamiliar with model categories; while I will be using some terminology lifted from algebraic topology, the proofs will be self-contained and not involve any Quillen theory.

References

[Joy02] André Joyal. Quasi-categories and kan complexes. *Journal of Pure and Applied Algebra*, 175(1-3):207–222, 2002.

[Lur03] Jacob Lurie. On ∞-topoi. preprint, 2003.

[Sim96] Carlos Simpson. Algebraic (geometric) n-stacks. preprint, 1996.