Semistable Reduction of overconvergent F-isocrystals

II: Semistable reduction Problem

We will keep the notation from the previous talk. In this talk we will discuss Kedlaya's approach to the semistable reduction conjecture of F-isocrystals ([5, 6, 7, 8]).

2.1 Statement of the Semistable Reduction Conjecture

Notation 2.1.1. Let k be a **perfect** field of characteristic p. K is the fraction field of the Witt vectors $\mathcal{O}_K = W(k)$ of k. Let $\mathcal{R}_K$ be the Robba ring over K.

Let us first recall the standard p-adic local monodromy conjecture. (see [9] or [10, Theorem 7.2.5])

Theorem 2.1.2. *Let $\mathcal{F}$ be a (σ, ∇)-module on over $\mathcal{R}_K$. Then $\mathcal{F}$ is quasi-unipotent, i.e. after pulling back along a finite étale map, $\mathcal{F}$ becomes unipotent.*

The aim of semistable reduction conjecture is to generalize this theorem to higher dimensional case, where one expect to replace finite étale map by an alteration. However, the quasi-unipotence is too strong a condition to be proved. Indeed, Kedlaya, in [5], interpreted the local unipotence in terms of the logarithmic extension. We will explain that in more detail in next subsection.

Log-isocrystal

We first gave the setup for the F-log-isocrystals. This construction is due to the work of Shiho [13, 14]. However, rather than going into the definition of log-scheme, we gave an intuitive explanation of this concept.

Definition 2.1.3. Let V be a (quasi-affinoid) rigid space over another rigid space W and $x_1, \cdots, x_n \in \Gamma(V, \mathcal{O})$ whose zero loci are smooth and meet transversely. Then $\Omega^{1,\log}_{V/W} \stackrel{\text{def}}{=} \Omega^1_{V/W} + \sum \mathcal{O}_V \frac{dx_i}{x_i}$. A log-$\nabla$-module is a locally free coherent module $\mathcal{F}$ over V together with a connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^{1,\log}_{V/W}$. The residue of $\mathcal{F}$ along the zero locus $V(x_i)$ of x_i is defined to be the endomorphism of $\mathcal{F}|_{V(x_i)}$ given by

$$\begin{array}{ccccc}
\mathcal{F} & \xrightarrow{\nabla} & \mathcal{F} \otimes \Omega^{1,\log}_{V/W} & \longrightarrow & \mathcal{F} \otimes \frac{dx_i}{x_i}\mathcal{O}_V \\
\downarrow & & & & \downarrow \\
\mathcal{F}|_{V(x_i)} & & \xrightarrow{\text{res.}} & & \mathcal{F}|_{V(x_i)}
\end{array}$$

Hypothesis 2.1.4. Recall the notations from 1.1.2, we further assume that $Y = P_k$ and $D = Y \backslash X$ is a simple normal crossing divisor. Assume further that there exists $x_1, \cdots, x_n \in \Gamma(P, \mathcal{O})$ whose reduction on Y defines the divisor D, i.e., $D = \cup V(\overline{x_i})$.

Remark 2.1.5. It is shown in [13, 14] that one can also talk about overconvergent log-isocrystals. The whole theory works well as in Berthelot's theory of overconvergent isocrystal. The definition of overconvergence of log-isocrystal is to similarly pull back to the diagonal embedding of X in $P \times P$, but one should be more careful about the log structure. I will not spend time on that. For details, one can consult Shiho's paper [13, 14] or [5, Section 6].

Definition 2.1.6. We say that an overconvergent isocrystal $\mathcal{F}$ on X extends to a log-isocrystal on Y, if $\mathcal{F}$ admits an extension to Y together with an integrable connection defined as in 2.1.3, such that the residue map along each of the divisor $V(x_i)$ is **nilpotent**.

Alteration

Now, we define the alteration. For details, one can consult [5, Section 3.1] or [4, Theorem 4.1].

Definition 2.1.7. For an irreducible k-variety X, an **alteration** is a proper dominant map $f : X' \to X$ with X' irreducible and f generically finite étale.

Theorem 2.1.8 (alteration)**.** *Let X be an irreducible k-variety and Z a proper closed subset of X. Then there exists an alteration $X' \to X$ such that X' admits a projective smooth compactification $\overline{X'}$ and $\overline{X'} \backslash X'$ is a simple normal crossing divisor.*

Conjecture 2.1.9 (Semistable Reduction Conjecture)**.** *Let k be a perfect field and $\mathcal{F}$ an overconvergent F-isocrystal on a k-variety X. Then after an alteration $\pi : X' \to X$, with $\bar{X}' \backslash X'$ a simple normal crossing divisor, $\pi^* \mathcal{F}$ can be extended to a log-isocrystal on X' with logarithmic poles along $\bar{X}' \backslash X'$.*

2.2 Unipotence Versus Logarithmic Extension

Before going further, let us first clarify the relationship between unipotence and logarithmic extension. This indicates the generalization from Theorem 2.1.2 to Conjecture 2.1.9.

Definition 2.2.1. Recall the setup from 2.1.3. Let I be an interval in $[0, +\infty)$. Consider $V \times A_K^n(I)$ over W, where the second factor has coordinates $t_1, \cdots, t_n$. Let $\mathrm{LNM}_{V \times A_K^n(I)/W}$ denote the category of log-∇-modules on $V \times A_K^n(I)$ relative to W with respect to $t_1, \cdots, t_n$, such that the residue along each $V(t_i)$ is **nilpotent** for $i = 1, \cdots, n$.

We say that a ∇-module $\mathcal{F} \in \mathrm{LNM}_{V \times A_K^n(I)/W}$ has **constant monodromy** if it is the pull back of a ∇-module on V. We say a that ∇-module has **unipotent monodromy** if it admits a filtration whose subquotients have constant monodromy. Let $\mathrm{ULNM}_{V \times A_K^n(I)/W}$ denote all the log-∇-modules on $V \times A_K^n(I)$ relative to W with respect to $t_1, \cdots, t_n$.

Theorem 2.2.2. *Let $\mathcal{F} \in \mathrm{LNM}_{V\times A^n_K(a,1)/K}$ be a* convergent *∇-module. Then it extends to a log-∇-module on $V \times A^n_K[0,1)$ if and only if $\mathcal{F}$ has unipotent monodromy. Moreover, this extension is unique if it exists and $\mathcal{F}$ is constant if and only if all the residues are zero.*

Proof. This is [5, Proposition 3.6.9]. The proof consists of the following ingredients.

(1) The extension implies the unipotence because of the convergence condition on $\mathcal{F}$. This is always satisfied in the case of overconvergent isocrystals where the norm of ∇ is bounded by the overconvergence as it should give a Taylor isomorphism $p_1^*\mathcal{F} \simeq p_2^*\mathcal{F}$. Then, roughly speaking, start from a section $s \in \Gamma(V \times A^n_K[0,1), \mathcal{F})$, we know that $\sum \frac{1}{I!}\partial^I s \cdot t^I$ is a horizontal section converging on the whole polydisc. We then quotient out this section and proceed the same work and finally prove the unipotence.

(2) Conversely, we use the following key lemma coming from cohomology computation:

Lemma 2.2.3. *If I is an open interval of $(0,1)$ or interval of the form $[0,a)$ with $a < 1$, there exists an equivalence of categories $\mathrm{ULNM}_{V\times A^n_K[0,0]/W} \to \mathrm{ULNM}_{V\times A^n_K(I)/W}$, where $\mathrm{ULNM}_{V\times A^n_K[0,0]/W}$ means overconvergence modules on V together with n commutative nilpotent operators.*

□

Using this theorem, one can freely translate between unipotence and logarithmic extension. In particular, the unipotence is easy to work with as it is just extension of constant modules. In contrast, the logarithmic extension is a global concept that behave well functorially.

Generalization Versus Unipotence

The following theorem [5, Theorem 3.4.3] is an interesting phenomena that the unipotence is determine only by the generic fiber.

Theorem 2.2.4. *Let I is an open interval of $(0,1)$ or interval of the form $[0,a)$ with $a < 1$. Let A be an integral affinoid K-algebra and $V = \mathrm{Max}(A)$. Let L be a complete archimedean field containing A (typically $(\mathrm{Frac}A)^\wedge$). Let $\mathcal{E}$ be a ∇-module over $V \times A^n_K(I)$, and $\mathcal{F} = \mathcal{E}\widehat{\otimes}_A L$. Then, $\mathcal{F}$ is constant (unipotent)* ***if and only if*** *$\mathcal{E}$ is constant (unipotent).*

Corollary 2.2.5. *The semistable reduction on smooth variety is insensitive to codimension 2 locus.*

Proof. The theorem is true essentially because one can use Taylor series to find horizontal sections and the convergence of Taylor series depends on the norm. The corollary follows by passing to the generic point of the divisor. □

Local to Global

The first step of going from local to global is to observe that the tube of an irreducible smooth divisor $D = V(\bar{f})$ in X looks like $Q \times A^1_K[0,1)$, where Q is $V(f)$ on P. Thus, an isocrystal on X gave a ∇-module $\mathcal{F}$ on $Q \times A^1_K(\epsilon, 1]$. We use the thin piece $Q \times A^1_K(\epsilon, 1)$ to talk about monodromy along Z.

Thus, according to Theorem 2.2.2, if $\mathcal{F}$ has unipotent monodromy along Z then we can extend $\mathcal{F}$ to a log-∇-module over X.

There is a subtlety here: say we are in the situation of $X = \mathbb{A}^2$ and $D = (\mathbb{A}^1_x \cup \mathbb{A}^1_y)$. We begin with an overconvergent ∇-module over $X \backslash D$. We know that **on $X\backslash\mathbb{A}^1_x$, $\mathcal{F}$ has unipotent monodromy along** $\mathbb{A}^1_y$, so we can extent $\mathcal{F}$ to a log-∇-module over $]X\backslash\mathbb{A}^1_x[$. However, apriori, we do not know that the extended thing is still overconvergent to the area of $](0,0)[$. One has to do some work to solve this problem and the answer is of course affirmative.

Remark 2.2.6. Up to now, we have not use Frobenius yet. The equivalence between unipotence and logarithmic extension works without Frobenius.

2.3 Valuational Approach

One of the reason that Kedlaya abandoned the approach by discussing every divisor separately is because if, along one divisor, it requires to do some finite étale extension, then he had no control on the ramification along other divisors.

Riemann-Zariski Space

Definition 2.3.1. Let F be a field finitely generated over k, then any k-valuation ($v(k) = 0$) is of the form $v : F^\times \to \mathbb{R}^n$ where $\mathbb{R}^n$ is endowed with the lexicographic order.

A valuation coming from an irreducible smooth divisor is called **divisorial** valuation.

The minimal n is called the **rank** (or **height**) of v.

If v is of height 1 and the residue field of v is algebraic over k, then v is called **minimal**.

Two valuations v_1, v_2 are considered equivalent if $v_1(x) > 0 \Leftrightarrow v_2(x)$, $\forall x \in F^\times$.

Definition 2.3.2. The Riemann-Zariski space T_F is the space of all equivalent classes of valuations on F. It has two topology generated by the following subsets as base:

(1) Zariski: $\{v \in T_F | v(x_1) \geq 0, \cdots, v(x_n) \geq 0\}$, for all $x_1, \cdots, x_n \in F^\times$.

(2) Patch: $\{v \in T_F | v(x_1) \geq 0, \cdots, v(x_n) \geq 0; v(y_1) > 0, \cdots v(y_m) > 0\}$, for all $x_1, \cdots, x_n, y_1, \cdots, y_m \in F^\times$.

We will use the Patch topology later on.

Theorem 2.3.3. *T_F is Hausdorff and compact with respect to the Patch topology and hence quasi-compact with respect to the Zariski topology.*

Definition 2.3.4. If $F = k(X)$, we say that v is **centered** on X if there exists a point $x \in X$, such that $v(\mathcal{O}_{X,x}) \geq 0$, or equivalently, $v(\mathcal{O}_{X,x}) \subseteq \mathfrak{o}_v$.

Semistable Reduction at a Valuation

Definition 2.3.5. Let v be a valuation on $k(X)$ and $\mathcal{F}$ an overconvergent isocrystal on X, we say that $\mathcal{F}$ has a **semistable reduction at** v if there exists an alteration $f : X' \to X$ and a compactification $X' \hookrightarrow \overline{X'}$ such that $f^*\mathcal{F}$ extends to a log-∇-module on a neighborhood $X' \subset V$ in $\overline{X'}$ and any extension of v to $F' = k(X')$ is **centered on** V.

Theorem 2.3.6. *To prove the Semistable Reduction Conjecture 2.1.9, it is enough to prove the semistable reduction for all valuations* $v \in T_{k(X)}$.

Proof. The key fact is that, for any V as given in 2.3.5, the valuations that centered on V form an open subset of $T_{k(X')}$, and hence, it proves the semistable reduction at an open subset of $T_{k(X)}$ (because $k(X')/k(X)$ is a finite separable extension).

Thus, use the compactness of the Riemann-Zariski spaces, we can prove the theorem in a way similar to the proof of Chow Lemma. Indeed, we need to show that $\mathcal{F}$ extends across all the missing divisors "generically" (see Theorem 2.2.4). □

Minimal Valuation

Theorem 2.3.7. *It is enough to prove the Semistable Reduction Conjecture 2.1.9 at minimal valuation.*

Proof. The reduction to the height 1 case is dealt in [6, Section 4.2]. Here, **we first time use the Frobenius.**

The reduction to the case when the residue field is algebraic over k is dealt in [6, Section 4.3]. This is essentially using Theorem 2.2.4. □