Semistable Reduction of overconvergent F-isocrystals

I: Isocrystals and Rigid Cohomology

The aim of this talk is to give a necessary background of isocrystals and rigid geometry in order to understand the semistable reduction conjecture. I will also do a brief literature review of this field.

1.1 Tubes and Strict Neighborhoods

This subsection is just a short version of [2, Section 1]. For details and proofs, one can consult the original paper.

Notation 1.1.1. Let K be a complete non-archimedean field (not necessarily discrete valued) of characteristic 0. Let $\mathcal{O}_K$ and k denote its ring of integers and residue field respectively. Assume that k has characteristic $p > 0$.

Notation 1.1.2. By a k-variety, I meant a reduced (not necessarily irreducible) separated scheme of finite type over k. (It could be shown that the theory only depends on the reduced scheme structure.) Through out the talk, X will be an open subscheme of a k-variety Y and $Z = X\backslash Y$ is the complement with the reduced scheme structure. P will always denote a topologically finite type formal scheme over $\mathcal{O}_K$ with a closed immersion $Y \hookrightarrow P_k$ into its special fiber. The generic fiber of P is the rigid analytic space P_K we are going to work on. Moreover, we require that P is **smooth** over $\operatorname{Spf} \mathcal{O}_K$ in an open neighborhood of X. To sum up, the following picture will show up very often.

$$\begin{tikzcd}
X \arrow[r, hook, "\text{open}"] \arrow[dr] & Y \arrow[r, hook, "\text{closed}"] \arrow[d] & P_k \arrow[r, hook, "\text{sp.fiber}"] \arrow[dl] & P \arrow[d] & P_K \arrow[l, "\text{gen.fiber}"'] \arrow[d] \\
 & \operatorname{Spec} k \arrow[rr, hook] & & \operatorname{Spf}(\mathcal{O}_K) & \operatorname{Sp}(K) \arrow[l]
\end{tikzcd} \tag{1.1.3}$$

Definition 1.1.4. A triple (X, Y, P) satisfying the conditions in Notation 1.1.2 is called a **frame**. A morphism between frames is a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, hook, "i'"] \arrow[d, "\pi"] & Y' \arrow[r, hook, "j'"] \arrow[d, "\pi"] & P' \arrow[d, "\pi"] \\
X \arrow[r, hook, "i"] & Y \arrow[r, hook, "j"] & P
\end{tikzcd} \tag{1.1.5}$$

such that π is **smooth** in an open neighborhood of X'.

Given a formal model of a rigid analytic space, we can write down a specialization map $\mathrm{sp} : P_K \to P_k$ surjective onto closed points of P_k. (A deep result of Raynaud is that knowing the specialization map is equivalent to specifying a formal model.)

Definition 1.1.6. Let $S \subset P_k$ be a subscheme of P_k. The **tube** of S in P is $\mathrm{sp}^{-1}(S)$, denoted by $]S[_P$.

The following proposition gives some visualization of tubes when $P = \mathrm{Spf}\, A$ is affine.

Proposition 1.1.7. *If $S \subset P_k$ is a closed immersion defined by $I = (\bar{f}_1, \cdots, \bar{f}_n)$, then $]S[_P = \{x \in P_K ||f_i(x)| < 1\}$, where f_i are liftings of $\bar{f}_i$ in A. If $S = \mathrm{Spec}(A_k)_{\bar{f}} \subset P_k$ is an open subscheme of P_k, then $]S[_P = \{x \in P_K ||f(x)| = 1\}$, where f is a lifting of $\bar{f}$ in A. Moreover, the descriptions do not depend on choice of the liftings.*

Notation 1.1.8. Now, assume for a moment that P and hence Y is affine. Say, $P = \mathrm{Spf}\, A$ and Y is cut out by $\bar{f}_1, \cdots, \bar{f}_n \in A_k$ with f_i their liftings to A. Moreover, Z is a closed subscheme of Y defined by $\bar{g}_1, \cdots, \bar{g}_m \in \mathcal{O}_Y$ with g_j their liftings to A. Let $\eta, \lambda \in (0, 1)$. Denote

$$[Y]_{P,\eta} = \{x \in P_K : |f_i(x)| \leq \eta\} \subset]Y[_P$$
$$U_\lambda = \{x \in]Y[_P : |g_j(x)| \geq \lambda\} \subset]Y[_P;$$
$$V_{\eta,\lambda} = [Y]_{P,\eta} \cap U_\lambda = \{x \in P_K : |f_i(x)| \leq \eta, |g_j(x)| \geq \lambda\}$$

Definition 1.1.9. A neighborhood V of $]X[_P$ in $]Y[_P$ is called a **strict neighborhood** if it satisfies the following equivalent conditions:

(1) $\{V,]Z[_P\}$ is an admissible covering of $]Y[_P$.

(2) For any $\eta \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that $V_{\eta,\lambda} \subseteq V \cap [Y]_{P,\eta}$.

(3) There exists sequence $\eta_n, \lambda_n \to 1^-$ as $n \to \infty$, such that V contains a standard strict neighborhood $V_{\underline{\eta},\underline{\lambda}} = \cup_n V_{\eta_n,\lambda_n}$.

The equivalence of above definitions is checked in [2, Section 1.2]. Moreover, the first definition can be generalized to non-affine case.

Exercise 1.1.10. Let $P = \mathrm{Spf}\, \mathcal{O}_K\langle x, y\rangle$, $Y = \mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$ the x-axis, and $X = Y \backslash \{0\}$. Draw a picture for a standard strict neighborhood of $]X[_P$ in $]Y[_P$.

The following theorem [2, Théorème 1.3.7] is one of the most crucial techniques in the theory of rigid cohomology. (see also [4, Proposition 2.2.9])

Theorem 1.1.11 (**Strong Fibration Theorem**). *Let π be a morphism of frames $(X, Y', P') \to (X, Y, P)$ inducing identity on X. $\bar{X}$ is the closure of X in $P'_Y = Y \times_P P'$ and suppose that $\pi : \bar{X} \to Y$ is proper.*

$$\begin{tikzcd}
X \arrow[r, hook, "i'"] \arrow[d, equal] & Y' \arrow[r, hook, "j'"] \arrow[d, "\text{``proper''}"] & P' \arrow[d, "\text{``smooth''}"] \\
X \arrow[r, hook, "i"] & Y \arrow[r, hook, "j"] & P
\end{tikzcd}$$

Let $\mathcal{I}' \subset \mathcal{O}_{P'}$ be the defining ideal of Y' in P', and let $\bar{\mathcal{I}}'$ be the defining ideal of Y' within P'_Y; suppose further that there exist sections $t_1, \cdots, t_d \in \Gamma(P', \mathcal{I}')$ whose reductions induce a basis of the conormal sheaf $\bar{\mathcal{I}}'/(\bar{\mathcal{I}}')^2$ on X. Put

$$P'' = P \times_{\mathcal{O}_K} \widehat{\mathbb{A}^d_{\mathcal{O}_K}} = \mathrm{Spf}\, \mathcal{O}_P\langle t_1, \cdots, t_d\rangle;$$

then the morphism $\phi : P' \to P''$ induces an isomorphism of some strict neighborhood of $]X[_{P'}$ within $]Y[_{P'}$ with some strict neighborhood of $]X[_{P''}$ within $]Y[_{P''}$.

Indeed, one does not have to remember the exact formulation of the theorem. Instead, one need to remember the following typical application of the theorem.

Example 1.1.12. We consider for the diagonal embedding with same Y. We require P to be smooth of dimension d and $\Gamma(P, \Omega^1_P)$ has a basis $dt_1, \cdots, dt_d$ for $t_i \in \mathcal{O}_P$:

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, equal] & Y \arrow[r, hook, "\Delta_j"] \arrow[d, equal] & P \times P \arrow[d, "\mathrm{pr}_1"] \\
X \arrow[r, hook, "i"] & Y \arrow[r, hook, "j"] & P
\end{tikzcd}$$

The strict neighborhood of $]X[_{P\times P}$ in $]Y[_{P\times P}$ is isomorphic to a strict neighborhood of $]X[_{P\times\mathbb{A}^d}=]X[_P\times A^d_K[0,1)$ in $]Y[_{P\times\mathbb{A}^d}=]Y[_P\times A^d_K[0,1)$ considered as embedding $P \hookrightarrow P \times \mathbb{A}^d$ using the zero section, where $A^d_K[0,1)$ is the standard notion of d-dimensional open unit polydisc with coordinates $t_1 - t'_1, \cdots, t_d - t'_d$.

1.2 Isocrystals

I will not talk about the definition of Monsky-Washnitzer cohomology (see for example) and actually the theory of rigid cohomology has incorporate the theory of Monsky-Washnitzer cohomology as a special case.

Rigid cohomology is a hybrid of Monsky-Washnitzer cohomology and the crystalline cohomology. The Monsky-Washnitzer cohomology has Lefschetz trace formula and is intimately related to algebraic de Rham theory of characteristic p. But it has a big restriction because of its poor functoriality. It is very inconvenient to "globalize" and glue affine pieces. Moreover, another crucial obstruction in the theory of Monsky-Washnitzer cohomology is the smoothness assumption.

In contrast to the Monsky-Washnitzer cohomology, crystalline cohomology is a relatively well understood theory. It has meaningful p-torsion, good sheaf theory and finite dimensionality for proper k-varieties. But crystalline cohomology uses Grothendieck topology which makes it hard to operate and to do explicit computation. Moreover, it seems to me that one does not know any finite dimensional statement except for proper k-schemes.

The rigid cohomology was firstly introduced in [1]. It combined the idea of overconvergence from Monsky-Washnitzer cohomology and the lifting technique from crystalline cohomology. At the expense of inverting p, it built up a bridge linking these two cohomology theories and then extracted good properties from them. One of the biggest achievement of theory of rigid cohomology is to give a purely p-adic proof of Weil Conjecture ([7]).

In this subsection, we will summarize the definition and basic properties of isocrystals following the approach of [2, Chapter 2].

Notation 1.2.1. In order to simplify the notation, sheaves are always sheaves of $\mathcal{O}$-modules. The general abelian sheaves behave exactly the same except the notational complication.

We start with a strict neighborhood V of $]X[_P$ in $]Y[_P$ and a sheaf $\mathcal{F}$ over V. Then, for any strict neighborhood V' of $]X[_P$ in $]Y[_P$ contained in V, we have a canonical morphism $\mathcal{F} \to \alpha_{VV'*}\alpha^*_{VV'}\mathcal{F}$, where $\alpha_{VV'} : V' \to V$ is the natural inclusion. Define

$$j^\dagger_V\mathcal{F} \overset{\text{def}}{=} \varinjlim_{V' \subset V} \alpha_{VV'*}\alpha^*_{VV'}\mathcal{F} = \text{``}\mathcal{F} \otimes_{\mathcal{O}_V} (\cup_{V' \subset V}\mathcal{O}_{V'})\text{''} \tag{1.2.2}$$

Moreover, we denote $j^\dagger\mathcal{F} = \alpha_{V*}j^\dagger_V\mathcal{F}$, where $\alpha_V : V \to]Y[_P$ is the natural inclusion.

Proposition 1.2.3. $j^\dagger_V\mathcal{F}$ *satisfies the following properties:*

(1) *When taking the limit in 1.2.2, we need only to take the limit over all the standard strict neighborhood.*

(2) *Let V' be as above, then $\alpha_{VV'*}j^\dagger_{V'}\alpha^*_{VV'}\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. In particular, the definition of $j^\dagger$ does not depend on the choice of V.*

(3) *The canonical map $\mathcal{F} \otimes j^\dagger_V\mathcal{O}_V \to j^\dagger_V\mathcal{F}$ is an isomorphism.*

(4) *The map $\mathcal{F} \to j^\dagger_V\mathcal{F}$ is an epimorphism. Moreover, if $\mathcal{F}$ is already a $j^\dagger_V\mathcal{O}_V$-module, the map is an isomorphism.*

(5) *Let $\pi : (X', Y', P') \to (X, Y, P)$ be a morphism of frames together with strict neighborhood V (resp. V') of $]X[_P$ (resp. $]X'[_{P'}$). Assume that $\pi_K(V') \subset V$. Let $\mathcal{F}$ be a sheaf on V. Then we have a functorial morphism $\pi^*j^\dagger_V\mathcal{F} \to j'^\dagger_{V'}\pi^*\mathcal{F}$. It is an isomorphism if $\pi^{-1}(X) = X'$. In particular, the same is true for $\pi^*j^\dagger\mathcal{F} \to j'^\dagger\pi^*\mathcal{F}$.*

(6) *If $Y = P_k$ is smooth irreducible projective k-variety and X is an open affine subset of Y, then $\Gamma(P_K, j^\dagger\mathcal{O}_{P_K})$ is a Monsky-Washnitzer overconvergent algebra associated to X.*

Now, let us define isocrystals on X overconvergent along Z. First, assume that we have a model 1.1.3. Let $\mathcal{I}$ be the ideal of the diagonal embedding $\delta : P_K \hookrightarrow P_K \times P_K$. Denote $\mathcal{P}^n = \mathcal{O}_{P_K \times P_K}/\mathcal{I}^{n+1}$. Then we can view $j'^\dagger\mathcal{P}^n$ as a polynomial algebra over $j^\dagger\mathcal{O}_{]Y[_P}$ truncated at degree n, where j is the injection of $Y \hookrightarrow P_k$ and $j' = \delta \circ j : Y \hookrightarrow P_k \times P_k$. The isomorphisms are different if we use different projection π_i.

Proposition 1.2.4. *Let $\mathcal{E}$ be a coherent $\mathcal{O}_{]X[_P}$-module. The following data are equivalent:*

(1) *A connection on $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{]X[_P}} \Omega^1_{]X[_P}$, such that $\nabla \circ \nabla = 0$, (i.e., ∇ is integrable).*

(2) *A compatible system of $\mathcal{P}^n$ isomorphisms $\epsilon_n : \mathcal{P}^n \otimes_{\mathcal{O}_{]X[}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{]X[}} \mathcal{P}^n$ with $\epsilon_0 = \mathrm{id}$, whose pull-back to $P_K \times P_K \times P_K$ satisfies a cocycle condition. Here we use the convention that tensoring $\mathcal{P}^n$ on the right means using the left projection of $\mathcal{O}_{]X[_P} \to \mathcal{P}^n$ and similar for tensoring on the left.*

Definition 1.2.5. An **overconvergent (integrable) ∇-module** on $]X[_P$ overconvergent along $]Z[_P$ is a coherent (and hence locally free) $j^\dagger \mathcal{O}_{]Y[_P}$-module $\mathcal{E}$ such that there exists an isomorphism $\epsilon : \mathrm{pr}_1^* \mathcal{E} \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{E}$ on a strict neighborhood of X in $P \times P$ inducing the same isomorphism as ϵ_n modulo $j'^\dagger \mathcal{I}^n$ and identity if we pull it back using δ, i.e., $\delta^* j'^\dagger = j^\dagger \delta^*$.

Notation 1.2.6. For simplicity, we will omit saying "integrable" as all the ∇-modules we are interested in are integrable. Moreover, when X and Y are clear, we just simply say overconvergent ∇-modules.

Proposition 1.2.7. *We list several basic properties of overconvergent ∇-modules.*

(1) Being an overconvergent ∇-module can be checked both locally on X and locally on P.

(2) Overconvergent ∇-modules have tensor products and inner $\mathcal{H}om$'s, i.e., if $\mathcal{E}$ and $\mathcal{F}$ are overconvergent ∇-modules, then the same is $\mathcal{H}om(\mathcal{E}, \mathcal{F})$, where the connection is defined to be $\nabla(\phi) = \nabla_{\mathcal{F}} \circ \phi - \phi \circ \nabla_{\mathcal{E}}$.

The following theorem is crucial in the construction of overconvergent isocrystals.

Theorem 1.2.8. *The category of overconvergent ∇-modules depends only on X and partly on Y. Precisely,*

(1) [2, Proposition 2.2.17] *Given two morphisms of frames $\pi, \pi' : (X', Y', P') \to (X, Y, P)$ as in 1.1.5, if they agree on Y', then there exists a canonical isomorphism of functors $\epsilon_{\pi,\pi'} : \pi_K^* \to \pi_K'^*$ on overconvergent ∇-modules.*

(2) [2, Théorèm 2.3.1] *Given the following commutative diagram with π smooth in a neighborhood of X*

$$\begin{tikzcd}
 & & P' \arrow[d, "\pi"] \\
X \arrow[r, hook] & Y \arrow[ur] \arrow[r] & P
\end{tikzcd}$$

then π_K^* induces an equivalence of categories of overconvergent ∇-modules on $]X[_P$ overconvergent alone $]Y \backslash X[_P$ and overconvergent ∇-modules on $]X[_{P'}$ overconvergent alone $]Y \backslash X[_{P'}$.

(3) [2, Théorèm 2.3.5] *If we have a commutative diagram*

$$\begin{tikzcd}
 & Y' \arrow[r] \arrow[d] & P' \arrow[d, "\pi"] \\
X \arrow[ur, hook] \arrow[r, hook] & Y \arrow[r] & P
\end{tikzcd}$$

*such that $\pi|_{Y'}$ is **proper** and π is smooth on a neighborhood of X, then π_K^* induces an equivalence of categories of overconvergent ∇-modules on $]X[_P$ overconvergent alone $]Y \backslash X[_P$ and overconvergent ∇-modules on $]X[_{P'}$ overconvergent alone $]Y' \backslash X[_{P'}$.*

Construction 1.2.9. By previous theorem, we know that the category of overconvergent ∇-modules is independent of the choice of P.

Now, let $X \subset Y$ be as before. First assume that there is a smooth lifting P of Y. We define **the category of isocrystals on X overconvergent along $Y \backslash X$**, denoted by $\mathrm{Isoc}^\dagger(X, Y/K)$, to be a rule to associate each lifting P an overconvergent ∇-module overconvergent along $]Y\backslash X[_P$ compatible with the pull back map π_K^* described in Theorem 1.2.8. Each of the overconvergent ∇-module on P_K is called a **realization** of the isocrystal.

For general $X \subset Y$, we can work locally on Y. By gluing local open subsets, we can get the category of overconvergent isocrystals.

Moreover, for a variety X, assume that X has a compactification $\bar{X}$. we define $\mathrm{Isoc}^\dagger(X/K) = \mathrm{Isoc}^\dagger(X, \bar{X}/K)$. By Theorem 1.2.8(3), this definition does not depend on the choice of the compactification.

Remark 1.2.10. The difference between isocrystals and their realizations (or, ∇-modules) is a just psychological problem. When people say isocrystals, they tend to mean the whole family of ∇-modules although the realization is an equivalence of categories and there is no substantial difference between the two.

Definition 1.2.11. Suppose k has characteristic p. Let F be the absolute Frobenius on X and Y. An overconvergent F-isocrystal on X overconvergent along $Y\backslash X$ is an overconvergent isocrystal $\mathcal{F}$ equipped with an isomorphism $\phi : F^*\mathcal{F} \to \mathcal{F}$.

Remark 1.2.12. The definition seems to have a small ambiguity on which Frobenius lifting you use on P, the lifting of Y where you realize the isocrystal. Nevertheless, Theorem 1.2.8(1) tells us that for every lifting of Frobenius, ϕ gives a specific morphism of overconvergent ∇-modules $\phi : F^*\mathcal{F} \to \mathcal{F}$.

Notation 1.2.13. We will use $\mathcal{F}$ to denote overconvergent isocrystals with a Frobenius in order to distinguish it from $\mathcal{E}$.

1.3 Rigid Cohomology

Definition 1.3.1. Given an isocrystal on X overconvergent along $Z = Y\backslash X$. First, assume that we can find a realization $\mathcal{E}$ on P_K. Then, we define the cohomology $H^*_{\mathrm{rig}}(X/K, \mathcal{E})$ of the overconvergent isocrystal to be the hypercohomology of the de Rham complex $\mathbb{H}^*(]Y[_P, \mathcal{E} \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P})$. If Y does not have a realization, we have to use a Čech cohomology argument.

Remark 1.3.2. It can be shown that this definition does not depend on the choice of P. Indeed, for different choices P and P', by Strong Fibration Theorem 1.1.11, if we pull back along $P \times P'$ via diagonal embedding, then we ended up with comparing cohomology of $\mathcal{E}$ on some strict neighborhood V and $\mathrm{pr}^* \mathcal{E}$ on $V \times A^n_K[0, 1)$. An explicit calculation in this case showed that the two complexes are homotopic equivalence.

Definition 1.3.3. Let V be a strict neighborhood of $]X[_P$ in $]Y[_P$ and $\mathcal{E}$ an $\mathcal{O}_V$-sheaf on V. Define the subsheaf of **sections of $\mathcal{E}$ supported on** $]Z[_P$: $\Gamma^\dagger_{]Z[}\mathcal{E} = \mathrm{Ker}(\mathcal{E} \to j^\dagger_V \mathcal{E})$. By Proposition 1.2.3(3), we have an exact sequence

$$0 \to \Gamma^\dagger_{]Z[}\mathcal{E} \to \mathcal{E} \to j^\dagger_V\mathcal{E} \to 0.$$

The following lemma may give you some idea about how the $\Gamma^\dagger_{]Z[}$ functors work.

Lemma 1.3.4. *Let* X_1 *and* X_2 *be two open subschemes of* Y *with complement* Z_1 *and* Z_2 *respectively. Denote* $X = X_1 \cup X_2$, $Z = Z_1 \cap Z_2$, $X' = X_1 \cap X_2$, $Z' = Z_1 \cup Z_2$ *and* j_1, j_2, j' *the natural immersions of* X_1, X_2, X' *into* Y. *Let* V *be a strict neighborhood of* $]X[$ *in* $]Y[$ *and* $\mathcal{E}$ *be a sheaf on* V. *Then we have natural isomorphism of functors:*
(1) $j_1^\dagger \circ j_2^\dagger \simeq j_2^\dagger \circ j_1^\dagger \simeq j'^\dagger$.
(2) $\Gamma^\dagger_{]Z_1[} \circ \Gamma^\dagger_{]Z_2[} \simeq \Gamma^\dagger_{]Z_2[} \circ \Gamma^\dagger_{]Z_1[} \simeq \Gamma^\dagger_{]Z[}$.

Definition 1.3.5. Let $T \subset X$ be a closed subscheme, $X' = X \backslash T$ and $\mathcal{E}$ a realization of an isocrystal on $]Y[_P$. One can define $\Gamma^\dagger_{]T[}$ by transforming $\mathcal{E}$ into an isocrystal on X', i.e., taking $j^\dagger$ with respect to X'. Define the **rigid cohomology of $\mathcal{E}$ with support in** T to be $H^*_{T,\mathrm{rig}}(X, \mathcal{E}) \stackrel{\mathrm{def}}{=} \mathbb{H}^*(]Y[_P, \Gamma^\dagger_{]T[}\mathcal{E} \otimes \Omega^\bullet_{]Y[})$.

Compact supported cohomology can also be defined for overconvergent isocrystals.

Definition 1.3.6. Let V be a strict neighborhood of $]X[_P$ in $]Y[_P$. We have $\iota :]Z[_P \cap V \hookrightarrow V$. Denote $\underline{\Gamma}(\mathcal{E}) \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathcal{E} \to \iota_* \iota^* \mathcal{E})$. Then, by Theorem B, $\mathbb{R}\underline{\Gamma}(\mathcal{E})$ is isomorphic to the two term complex: $\mathcal{E} \to \iota^* \iota^* \mathcal{E}$. The **rigid cohomology with compact supports** is defined to be $H^i_{c,\mathrm{rig}}(X/K, \mathcal{E}) = \mathbb{H}^i(V, \mathbb{R}\underline{\Gamma}(\mathcal{E} \otimes \Omega^\bullet_V))$. (Similar to rigid cohomology, this definition does not depend on the choice of lifting P.)

Now, I will list a number of theorems regarding rigid cohomology. For more detailed, one can consult [6, Section 1].

Theorem 1.3.7. *Let* $\mathcal{F}$ *be an overconvergent* F*-isocrystal on a variety* X. *Then the rigid cohomology* $H^i_{\mathrm{rig}}(X, \mathcal{F})$ *and* $H^i_{c,\mathrm{rig}}(X, \mathcal{F})$ *are finite dimensional* K*-vector spaces for all* i.

Theorem 1.3.8 (Poincaré Duality). *Let* $\mathcal{F}$ *be an overconvergent* F*-isocrystal on a smooth variety* X *of pure dimension* d. *Then for any close subscheme* $T \subset X$, *there are natural perfect pairings*

$$H^i_{T,\mathrm{rig}}(X/K, \mathcal{F}) \times H^{2d-i}_{c,\mathrm{rig}}(T/K, \mathcal{F}^\vee) \to K$$

Theorem 1.3.9 (Künneth Formula). *Let* $\mathcal{F}_1/K$ *(resp.* $\mathcal{E}_F/K$*) be overconvergent* F*-isocrystals on a* k*-variety* X_1 *(resp.* X_2*). Put* $X = X_1 \times_k X_2$, *and* $\mathcal{F} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \stackrel{\mathrm{def}}{=} \mathrm{pr}_1^* \mathcal{F}_1 \otimes \mathrm{pr}_2^* \mathcal{F}_2$ *where* $\mathrm{pr}_i : X \to X_i$ *is the natural projection. Then there is a natural isomorphism*

$$\bigoplus_{j+l=i} H^j_{c,\mathrm{rig}}(X_1/K, \mathcal{F}_1) \otimes H^l_{c,\mathrm{rig}}(X_2/K, \mathcal{F}_2) \simeq H^i_{c,\mathrm{rig}}(X/K, \mathcal{F}).$$

Using the bridge of rigid cohomology, one can carry the properties from crystalline cohomology to Monsky-Washnitzer cohomology and hence prove the finite dimensionality of Monsky-Washnitzer cohomology [3, Corollaire 3.2].

As Kedlaya has mentioned in Arizona Winter School, the rigid cohomology has a good trace formula coming from Washnitzer-Monsky cohomology. Moreover, Kedlaya used rigid cohomology to prove Weil Conjecture purely p-adically [7]. As we are short of space, we will not discuss this in details.

Remark 1.3.10. This finite dimensionality of rigid cohomology of k-varieties is firstly proved by Berthelot [3, Théorèm 3.1] for X smooth and $\mathcal{E} = \mathcal{O}_X$. Berthelot's proof consists of four important ingredients.

(1) A long exact sequence for devisage ([3, Proposition 2.5]):

$$\cdots \to H^i_T(X/K) \to H^i_S(X/K) \to H^i_{S'}(X'/K) \to \cdots$$

where $T \subset S \subset X$ are closed subschemes and $S' = S \backslash T$, $X' = X \backslash T$.

(2) Let $f : X' \to X$ be a (surjective) finite flat morphism. To check the finite dimensionality of $\mathcal{E}$ over X, it is suffice to check it for $f^*\mathcal{E}$ over X'.

(3) Gysin's isomorphism ([3, Corollaire 5.6], more generally, see [8, Theorem 4.1.1]):

$$H^i_{T,\mathrm{rig}}(X/K) \simeq H^{i-2c}_{\mathrm{rig}}(T/K)$$

for smooth pair (T, X).

(4) Comparison theorems to crystalline cohomology in smooth proper case:

$$H^i_{\mathrm{rig}}(X/K) = H^i_{\mathrm{cris}}(X, W(k)) \otimes K$$

(Of course, in terms of finite dimensionality, this could be replaced by Kiehl's finiteness theorem.)

The same strategy can not be used to prove the general finiteness theorem 1.3.7 because an isocrystal usually does not extend to a proper X. In [6], Kedlaya adopted an indirect way to approach the problem using his p-adic local monodromy theorem [5] and fibration by curves.

But one should expect a more direct approach to the problem using Berthelot's strategy. To this end, one need the following semistable reduction conjecture. The terminology in the theorem will be explained in later talks.

Theorem 1.3.11. *Let $\mathcal{F}$ be an overconvergent F-isocrystal on a k-variety X. Then after an alteration $\pi : X' \to X$, with $\bar{X}'\backslash X'$ a simple normal crossing divisor, $\pi^*\mathcal{F}$ can be extended to a log-isocrystal on X' with logarithmic poles along $\bar{X}'\backslash X'$.*

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