Ramification Theory for Local Field with Imperfect Residue Field

Liang Xiao

1 Introduction

The ramification of a complete discrete valuation field has been studied since the years of Hilbert, who introduced the notion of higher ramification groups. Later, Artin, Hasse, Tate, and other people made extraordinary contribution to generalize Hilbert's idea. A good reference for the classical ramification theory is Serre's fantastic book [9].

Notation 1.1. Let l/k be a finite Galois extension of complete discrete valuation fields. Let $\mathcal{O}_k$, $\mathcal{O}_l$, π_k, π_l, $\bar{k}$, and $\bar{l}$ be rings of integers, uniformizers and residue fields, respectively. Let $G = G_{l/k}$ be the Galois group. Define $v_l(\cdot)$ to be the valuation on l. Let $e = v_l(\pi_k)$ be the *naïve ramification degree.*

Hypothesis 1.2. In this section, we assume that the residue field k is perfect.

Definition 1.3. The most natural way to define higher ramification subgroups of the Galois group G is due by Hilbert:

$$g \in G_a \text{ if and only if } v_l(gx - x) \geq a + 1, \forall x \in \mathcal{O}_l.$$

Indeed, $G_{-1} = G$, $G_0 = I$ is the inertia subgroup, and $G_1 = W$ is the wild inertia subgroup.

However, there is a disadvantage of this. Namely, it does not respect quotient and hence it does not give a filtration on the absolute Galois group G_k. Herbrand defined an ad hoc looking function ϕ and gave G an upper numbering filtration, which does extend to G_k. (We will give a working definition later.)

Definition 1.4. We set $G_u = G_{\lceil u \rceil}$ for $u \in [-1, \infty)$. Define

$$\phi(u) = \int_0^u \frac{dt}{[G_0 : G_t]}.$$

We set $G^{\phi(u)} = G_u$.

Proposition 1.5. *It turns out, if H is a normal subgroup of G, then $(G/H)^v = G^v H/H$ for all v. Hence the upper numbering filtration patches to give a filtration $\mathrm{Fil}^v G_k$ on G_k.*

Definition 1.6. Let $\rho : G_k \to \mathrm{GL}(V)$ be a representation of finite local monodromy (the image of inertia is finite), where V is a finite dimensional vector space over a characteristic 0 field. Define the *Artin conductor*

$$\mathrm{Art}(\rho) \overset{\mathrm{def}}{=} \sum_{a\in\mathbb{Q}_{\geq 0}} a\cdot \dim\left(V^{\mathrm{Fil}^{a+}G_k}/V^{\mathrm{Fil}^{a}G_k}\right), \tag{1.7}$$

where $\mathrm{Fil}^{a+}G_k = \overline{\cup_{b>a}\mathrm{Fil}^b G_k}$.

One can also define the *Swan conductor*

$$\mathrm{Swan}(\rho) \overset{\mathrm{def}}{=} \sum_{a\in\mathbb{Q}_{\geq 1}} (a-1)\cdot \dim\left(V^{\mathrm{Fil}^{a+}G_k}/V^{\mathrm{Fil}^{a}G_k}\right), \tag{1.8}$$

which measures the wild ramification of ρ.

Theorem 1.9 (Hasse-Arf Theorem)**.** *The conductors* $\mathrm{Art}(\rho)$ *and* $\mathrm{Swan}(\rho)$ *are non-negative integers.*

In practise, we will consider irreducible representation ρ which exactly factors through a finite Galois extension l/k. In this case, let $b(l/k) = \max\left\{b|G^b_{l/k} \neq \{1\}\right\}$ and then we have

$$\mathrm{Art}(\rho) = b(l/k)\cdot \dim\rho. \tag{1.10}$$

This $b(l/k)$ is called the *highest ramification break* of l/k.

$$b(l/k)\begin{cases} = 0 & \text{If } l/k \text{ is unramified} \\ = 1 & \text{If } l/k \text{ is tamely ramified} \\ > 1 & \text{If } l/k \text{ is wildly ramified.}\end{cases} \tag{1.11}$$

Proposition 1.12. *If $\mathcal{O}_l/\mathcal{O}_k$ is generated by one element x, we have an explicit formula for $b_{l/k}$*

$$b_{l/k} = \frac{1}{e}\Big(\sum_{1\neq g\in G} v_l(gx-x) + \max_{1\neq g\in G} v_l(gx-x)\Big).$$

Alternative Interpretation via Rigid Space

One of the main idea of Abbes-Saito's construction comes from a reinterpretation of the number $b(l/k)$ via geometric connected components of certain rigid space.

The Hypothesis 1.2 implies that $\mathcal{O}_l/\mathcal{O}_k$ is generated by one element x in $\mathcal{O}_l$. Let $P(x) = 0$ be its minimal polynomial.

Notation 1.13. We set $|\pi_k| = \theta$ as this number will come out a lot later.

Proposition 1.14. *The rigid space $X = \{u||u| \leq 1, |P(u)| < \theta^a\}$ has $[l:k]$ geometric connected components if and only if $a \geq b(l/k)$.*

Proof. A strict proof can be found at [5, Lemme 2.4] or [1, Lemma 6.6]. We will give a rough idea of why this is true.

The picture here is that if a is very large, we restrict u to be very close to the roots of $P(u) = 0$, the conjugates of x. The rigid space X should be geometrically disjoint union of very small discs centered at each of conjugates of x. When a becomes smaller, the discs become larger and, at some point, some of them crash into one disc, which decreases the number of geometric connected components.

The cut-off condition is obviously $|u - x| < \max_{1 \neq g \in G} |gx - x|$. Note that $P(u) = \prod_{g \in G}(u - gx)$. Hence, one has $|u - gx| = |gx - x|$. Thus,

$$|P(u)| = \prod_{g \in G} |u - gx| = |u - x| \prod_{1 \neq g \in G} |x - gx| < \theta^{b(l/k)}.$$

If one stares at this explanation for a moment, one may realize that this can be actually turned into a rigorous proof. □

2 Abbes-Saito Ramification Filtrations

From now on, we drop the Hypothesis 1.2 and consider the case when the residue field is imperfect. The whole theory is initiated by Kato in [6] for abelian representations and by Abbes and Saito in [1] for general case. Kedlaya [7] gave a different approach for equal characteristic case following the ideas of Christol, Matsuda, Mebkhout, and Tsuzuki.

When the residue field is imperfect, there are two kinds of wild ramifications. The first kind is typically given by an Eisenstein extension $y^e + \alpha_1 y^{e-1} + \cdots \alpha_e = 0$ with $p|e$. The second kind has imperfect residue field extension but does not change the group of valuations, for example, $x^p = \alpha$ for $\bar{\alpha}$ the image of α in $\bar{k}$ not a p-th power, i.e. $\bar{\alpha} \in \bar{k} \backslash \bar{k}^p$. Unfortunately, one can not separate these two kinds of ramifications like what we did for unramified extension or tamely ramified extension. After base change, the order of such sub-extensions may exchange.

Another difficulty in this case is that $\mathcal{O}_l/\mathcal{O}_k$ is no longer monogenic. Thus, the naïve generalization of Proposition 1.12 is not possible. However, the rigid geometric interpretation can be generalized to this case. This is carried out by Abbes and Saito in [1] and [2].

Definition 2.1. Take $Z = (z_j)_{j \in J} \subset \mathcal{O}_l$ to be a finite set of elements generating $\mathcal{O}_l$ over $\mathcal{O}_k$, i.e., $\mathcal{O}_k[(u_j)_{j \in J}]/\mathcal{I} \simeq \mathcal{O}_l$ mapping u_j to z_j. Let $(f_i)_{i=1,\cdots,n}$ be a finite set of generators of $\mathcal{I}$. Define the *Abbes-Saito space* to be

$$as^a_{l/k,Z} = \left\{ (u_J) \left| \begin{array}{cc} |u_j| \leq 1, & j \in J \\ |f_i(u_j)| \leq \theta^a, & 1 \leq i \leq n \end{array} \right. \right\}. \qquad (2.2)$$

We denote the *geometric* connected components of $as^a_{l/k,Z}$ by $\pi_0^{\mathrm{geom}}(as^a_{l/k,Z})$. The *highest ramification break* $b_{l/k}$ of the extension l/k is defined to be the minimal b such that $\forall a > b$, $\#\pi_0^{\mathrm{geom}}(as^a_{l/k,Z}) = [l : k]$.

They also define a version of logarithmic ramification number, which will give rise to Swan conductors. We will not introduce the general definition, rather we will give a working definition when l/k is totally ramified.

Lemma 2.3. *We can choose* $c_1, \cdots, c_m \in \mathcal{O}_l^\times$ *and* $b_1, \cdots, b_m \in \mathcal{O}_k^\times$ *such that* $\bar{b}_j = \bar{c}_j^{p^{r_j}}$ *in* $\bar{l}$ *for some* $r_j \in \mathbb{Z}_{\geq 0}$ *and all* $j = 1, \cdots, m$. *We also choose the uniformizer* π_l *so that* $\pi_k \equiv \pi_l^e \bmod \pi_l^{e+1}$. *As a consequence,* $c_1, \cdots, c_m, \pi_l$ *generate* $\mathcal{O}_l$ *over* $\mathcal{O}_k$. *More precisely,*

$$\left\{c_J^{e_J}\pi_l^i \,|\, e_j \in \{0, \cdots, p^{r_j}-1\} \text{ for all } j, \text{ and } i \in \{0, \cdots, e-1\}\right\} \tag{2.4}$$

form a basis of $\mathcal{O}_l$ *over* $\mathcal{O}_k$.

Let $\mathcal{O}_k[u_0, \cdots, u_m]/\mathcal{I} \xrightarrow{\sim} \mathcal{O}_l$ *be the map sending* u_j *to* c_j *for* $j = 1, \cdots, m$ *and* u_0 *to* π_l. *We will choose a set of generators* $p_0, \cdots, p_m$ *of* $\mathcal{I}$ *as follow: for every* $c_j^{p^{r_j}}$ *or* π_l^e, *one can write it in terms of the basis listed in 2.4. This will give us an element* p_j *or* p_0 *in* $\mathcal{I}$. *Obviously,* $p_0, \cdots, p_m$ *generate* $\mathcal{I}$. *Moreover,*

$$\begin{aligned} p_j &\in u_j^{p^{r_j}} - b_j + (u_0, \pi_k)\cdot k[\![u_0, \cdots, u_m]\!], \\ p_0 &\in u_0^e - \pi_k + (u_0\pi_k, \pi_k^2)\cdot k[\![u_0, \cdots, u_m]\!]. \end{aligned}$$

Definition 2.5. Keep the notation from Lemma 2.3. Define the standard *Abbes-Saito space* and *logarithmic Abbes-Saito space* to be

$$\begin{aligned} as^a_{l/k} &= \left\{(u_0, \cdots, u_m)\left| \begin{array}{c} |u_0| \leq 1, \cdots, |u_m| \leq 1, \\ |p_0(u_{J^+})| \leq \theta^a, \cdots, |p_m(u_{J^+})| \leq \theta^a \end{array}\right.\right\} \\ \text{and } as^a_{l/k,\log} &= \left\{(u_0, \cdots, u_m)\left| \begin{array}{c} |u_0| \leq 1, \cdots, |u_m| \leq 1, \\ |p_0(u_{J^+})| \leq \theta^{a+1}, |p_1(u_{J^+})| \leq \theta^a, \cdots, |p_m(u_{J^+})| \leq \theta^a \end{array}\right.\right\}, \end{aligned}$$

Similarly, the *highest logarithmic ramification break* $b_{l/k,\log}$ of the extension l/k is defined to be the minimal b such that $\forall a > b$, $\#\pi_0^{\mathrm{geom}}(as^a_{l/k,\log}) = [l : k]$.

Theorem 2.6. *The Abbes-Saito spaces have the following properties.*

(1) *The space* $as^a_{l/k,Z}$ *does not depend on the choice of generators* $(f_i)_{i=1,\cdots,n}$ *of* $\mathcal{I}$. *If we change for another generating sets of* Z, *then the geometric connected components* $\pi_0^{\mathrm{geom}}(as^a_{l/k,Z})$ *does not change. In particular, ramification breaks are well-defined* [1, Section 3]. *Moreover, the logarithmic ramification break* $b_{l/k,\log}$ *is also well-defined.*

(2) *The ramification break (resp. logarithmic ramification break) gives rise to a filtration of normal subgroups* $\mathrm{Fil}^a G_k$ *(resp.,* $\mathrm{Fil}^a_{\log} G_k$*) on the Galois group* G_k [1, Theorem 3.3, 3.11]. *Moreover, for* l/k *a finite Galois extension, both highest ramification breaks are rational numbers* [1, Theorem 3.8, 3.16].

(3) *Let* l/k *be a finite separable Galois extension. If* l/k *is unramified, then* $\mathrm{Fil}^a G_l = \mathrm{Fil}^a G_k$ [1, Proposition 3.7]. *If* l/k *is tamely ramified with ramification index* m, *then* $\mathrm{Fil}^{ma}_{\log} G_l = \mathrm{Fil}^a_{\log} G_k$ [1, Proposition 3.15].

(4) *Define* $\mathrm{Fil}^{a+}G_k = \overline{\cup_{b>a}\mathrm{Fil}^bG_k}$ *(resp.* $\mathrm{Fil}^{a+}_{\log}G_k = \overline{\cup_{b>a}\mathrm{Fil}^b_{\log}G_k}$*). Subquotients* $\mathrm{Fil}^aG_k/\mathrm{Fil}^{a+}G_k$ *(resp.* $\mathrm{Fil}^a_{\log}G_k/\mathrm{Fil}^{a+}_{\log}G_k$*) are p-abelian groups for any* $a \in \mathbb{Q}_{\geq 1}$ *(resp.* $a \in \mathbb{Q}_{\geq 0}$*) and are* 0 *if* $a \notin \mathbb{Q}$*, except possibly false in absolutely unramified and non-logarithmic case* ([1, Theorem 3.8, 3.16], [2, Theorem 1])*.*

(5) *The inertia subgroup is* Fil^aG_k *if* $a \in (0, 1]$ *and the wild inertia subgroup is* $\mathrm{Fil}^{1+}G_k = \mathrm{Fil}^{0+}_{\log}G_k$ [1, Theorem 3.7, 3.15]*.*

(6) *When the residue field* $\bar{k}$ *is perfect, arithmetic ramification filtrations agree with the classical upper numbered filtration in the following way:* $\mathrm{Fil}^aG_k = \mathrm{Fil}^{a-1}_{\log}G_k = G^a_k$ [1, Section 6.1]*.*

Proof. I will sketch some of the proofs.

(1) The first statement is straightforward by matching up points. For the second, one can show that if we add a new (dummy) generator in Z, the new Abbes-Saito space admits a fibration over the original Abbes-Saito space whose fibers are closed discs of radius θ^a.

(2) The first statement is just abstract nonsense. The second one is essentially because one defines the Abbes-Saito space over k and the geometric connect components and be seen over the algebraic closure k^{alg}, which has valued group in $|k^\times|^{\mathbb{Q}}$.

(3) For the unramified case, it is essentially because, for any finite Galois extension k'/k disjoint from l/k, $\mathcal{O}_{lk'} \simeq \mathcal{O}_l \otimes_{\mathcal{O}_k} \mathcal{O}_{k'}$. Thus, one can match up two Abbes-Saito spaces in a natural way. In the tamely ramified and logarithmic case, one can also naturally identify two Abbes-Saito spaces [1, Proposition 9.8].

(4) The proof used formal models and their reductions of Abbes-Saito spaces, which is the main theorem in [2]. Saito proved a stronger version of this in [8, Theorem 1.3.3] stating that the graded piece of logarithmic filtrations are actually killed by p.

(5) is an easy fact.

(6) follows from the explicit calculation in Proposition 1.12 and 1.14. □

One can define Artin conductors and Swan conductors as in classical case, using the same Formula 1.7, 1.8. Essentially, for an irreducible representation ρ which exactly factors through a Galois extension l/k, $\mathrm{Art}(\rho) = b(l/k) \cdot \dim \rho$ and $\mathrm{Swan}(\rho) = b_{\log}(l/k) \cdot \dim \rho$.

Who Cares about Imperfect Residue Field, anyway?

Temporarily, we do not use the notations about l/k setup earlier.

One of the main reason that people are interested in Swan conductor is the

Theorem 2.7 (Grothendieck-Ogg-Shavarevich Formula)**.** *Let* k *be a perfect field of characteristic* p*, prime to* l*. Let* X *be a smooth complete curve over* k *and* U *a dense open subset. Assume for simplicity,* $X\backslash U = \{x_1, \cdots, x_n\}$ *are all rational points over* k*. Given a lisse* $\mathbb{F}_l$*-sheaf* $\mathcal{F}$ *on* U*, one can read off Swan conductor* $\mathrm{Swan}_{x_j}(\mathcal{F})$ *at each of the missing point* x_j*. Then,*

$$\chi_c(U, \mathcal{F}) = \sum_{i=0}^{2}(-1)^i \dim_{\mathbb{F}_l} H^i_c(U, \mathcal{F}) = \chi_c(U, \mathbb{F}_l) \cdot \mathrm{rank}_{\mathbb{F}_l}\mathcal{F} - \sum_{j=0}^{n} \mathrm{Swan}_{x_j}\mathcal{F}. \tag{2.8}$$

There is an analogue result for overconvergent F-isocrystals.

Very vaguely speaking, one can view this as an analogy of Grothendieck-Riemann-Roch theorem. So, it is natural to ask for a higher dimensional Theorem 2.7.

Let X be a smooth complete variety over k and D a divisor with simple normal crossing. Let $\mathcal{F}$ be a lisse $\mathbb{F}_l$-sheaf on $X\backslash D$. The first difficulty is how to define Swan conductors $\mathrm{Swan}_{D_i}(\mathcal{F})$ along an irreducible components D_i of D. An obvious definition is to pass to the completion $R = \mathcal{O}^{\wedge}_{X,D_i}$ at the generic point of D_i. One immediately finds out that R is a complete discrete valuation ring with residue field the function field of D_i which is typically imperfect if $\dim D_i \geq 1$.

In some sense, the above definition gives a possible way to overcome this difficulty can move on. Indeed, Saito, in [8], proved a higher dimensional Grothendieck-Ogg-Shavarevich Formula under some technical conditions. However, we will not go into that in this talk.

3 Hasse-Arf Theorem

The main topic today is the following

Conjecture 3.1 (Hasse-Arf Theorem). *The conductors* $\mathrm{Art}(\rho)$ *and* $\mathrm{Swan}(\rho)$ *are non-negative integers. Moreover, the subquotients* $\mathrm{Fil}^a G_k/\mathrm{Fil}^{a+} G_k$ *and* $\mathrm{Fil}^a_{\log} G_k/\mathrm{Fil}^{a+}_{\log} G_k$ *are abelian groups killed by* p *for* $a > 1$ *and* 0*, respectively.*

In the case when k is of equal characteristic $p > 0$, there is another definition of Artin/Swan conductor using p-adic differential equations, which was first introduced by Christol, Matsuda, Mebkhout, and Tsuzuki in the classical case and was carried out in general case by Kedlaya [7] later.

Kedlaya proved Hasse-Arf theorem for the conductors he defined. Matsuda then asked if one can prove a comparison theorem between the Abbes-Saito's definition and Kedlaya's definition. Therefore, one can obtain a Hasse-Arf theorem for the ramification filtrations defined by Abbes and Saito.

The first step was carried out by Bruno Chiarellotto and Andrea Pulita [4] in rank 1 case. At the same time, I worked out the general case [10].

Proof. Since I do not have time to introduce Kedlaya's definition, I will just vaguely say how one can prove Conjecture 3.1 in equal characteristic case, hiding Kedlaya's definition in the proof. We will focus on the Artin conductors.

Let k be a complete discrete valuation field of equal characteristic $p > 0$. We may always reduce to the case when k has a finite p-basis, or equivalently, $\dim_{k^p}(k) < \infty$. The essential case is the study of a finite Galois totally ramified extension l/k whose wild ramification part is non-trivial. As in Lemma 2.3, the extension of ring of integers $\mathcal{O}_l/\mathcal{O}_k$ is generated by $c_1, \cdots, c_m, \pi_l$, or vaguely speaking, some "good" elements which allow us to write down explicit equations by which they generate $\mathcal{O}_l$.

<u>**Step I:**</u> we lift the Abbes-Saito space $as^a_{l/k}$ to a rigid space $AS^a_{l/k}$ over $A^1_K[\eta_0, 1)$ for $\eta_0 \to 1$, where K is the fraction field of a Cohen ring of $\bar{k}$ and $A^1_K[\eta_0, 1) = \{x | \eta_0 \leq |x| < 1\}$.

Roughly speaking, the lifting process is just writing down equations that define $as^a_{l/k}$ as an affinoid subspace of $A^{m+1}_k[0,1]$ and lifting the coefficients to $K[\![T]\!]$; then these equations will give the space $AS^a_{l/k}$. Vaguely speaking, one can show that geometric connected components of $as^a_{l/k}$ are in one-to-one correspondence with "geometric" connected components of $AS^a_{l/k}$ when $\eta_0 \to 1$. Here, "geometric" means up to a base extension from $A^1_K[\eta_0, 1)$ to $A^1_{K'}[\eta_0^{1/e_{k'/k}}, 1)$, where k' is a finite separable extension of k of naive ramification degree $e_{k'/k}$ and K' is the fraction field of a Cohen ring of the residue field of k'.

In short, this step lifts the Abbes-Saito space to a rigid space over a characteristic 0 field.

Step II: Kedlaya [7] constructs a differential equation $\mathcal{E}$ over $A^1_K[\eta_0, 1)$ for the field extension l/k. The essence of the construction is the fact that $A^1_L[\eta_0^{1/e}, 1)$ is finite and étale over $A^1_K[\eta_0, 1)$, where L is the fraction field of a Cohen ring of the residue field $\bar{l}$ of l. Hence, one can push-forward the constant differential module over $A^1_L[\eta_0^{1/e}, 1)$ and obtain a differential module over $A^1_K[\eta_0, 1)$.

In short, we can construct a differential equation over $A^1_K[\eta_0, 1)$ using the finiteness and the étaleness.

Step III: we consider the thickening space

$$TS^a_K = \cup_{\eta \in [\eta_0, 1)} A^1_K[\eta, 1) \times A^{m+1}_K[0, \eta]$$

by using the homomorphism $\tilde{\pi}$ to pull out directions corresponding to the p-basis. We proved that spectral norms on $\mathcal{F} = \tilde{\pi}^*\mathcal{E}$ over TS^a_K are the same as spectral norms on $\mathcal{E}$ over $A^1_K[\eta_0, 1)$ in an appropriate sense.

In short, we have a thickening process.

Step IV: In [10, Section 4.3], we proved a comparison theorem

$$A^1_L[\eta_0^{1/e}, 1) \times_{A^1_K[\eta_0, 1), \tilde{\pi}} TS^a_K \simeq AS^a_{l/k},$$

where $\tilde{\pi}$ is a twisted projection and $TS^a_K = \cup_{\eta \in [\eta_0, 1)} A^1_K[\eta, 1) \times A^{m+1}_K[0, \eta^a]$ is the thickening space. As a consequence, the question of "geometric" connected components of $AS^a_{l/k}$ can be interpreted as the question of "geometric" connected components of $A^1_L[\eta_0^{1/e}, 1) \times_{A^1_K[\eta_0, 1), \tilde{\pi}} TS^a_K$, which can be further reinterpreted by standard theory of p-adic differential equations as spectral norms on the differential equation $\mathcal{F}$ over TS^a_K.

In short, this step proves the $AS = TS$ theorem and links the geometric connected components with spectral norms on thickening space.

Step V: Combining previous four steps, the study of highest ramification break is transformed to the study of spectral norms of $\mathcal{E}$ over $A^1_K[\eta_0, 1)$. Inspired by Borger's work [3], in [7], Kedlaya used the trick of generic rotation to reduce to the case when the spectral norm of $\partial/\partial T$ dominates other spectral norms, where T is the coordinates of $A^1_K[\eta_0, 1)$. Then one can forget about all other differential operators coming from p-basis of $\bar{k}$, equivalently, one can add all the p^∞-th roots of the chosen p-basis into k without changing the (differential) ramification break. Hence, we are reduced to the classical case.

In short, this step shows that after generic rotation, the uniformizer direction is dominant and in this circumstance, one can perfectify the residue field without changing the highest ramification break. $\square$

4 Mixed Characteristic

This will be carried out in [11].

Now we explain how one operates a similar proof in the mixed characteristic case for non-logarithmic ramification filtration. Now, assume that k is of mixed characteristic $(0, p)$. For safety, we assume that $v_k(p) \gg 0$.

First of all, we do NOT need step I! The only reason that we have to lift a rigid space is to put it in characteristic 0 so that we can talk about differential equation and spectral norms. So, we just work with Abbes-Saito space itself as it is already over a characteristic 0 field k.

Step II needs a differential treatment. Pretend for a moment that we have a homomorphism $\psi : \mathcal{O}_k \to \mathcal{O}_k[\![\delta_0/\pi_k]\!]$, which sends π_k to $\pi_k + \delta_0$. Then,

$$D^a_{l/k} = \mathrm{Max}\big(l \times_{k,\psi} k\langle \pi_k^{-a}\delta_0\rangle\big) \to A^1_k[0, \theta^a]$$

is finite and étale will give the differential equation $\mathcal{E}$. Unfortunately, the map ψ NEVER exists for a mixed characteristic K because $\psi(p) = p$ if ψ were a homomorphism.

The trick is that we do NOT take ψ to be a homomorphism but only a map. Actually, it is better than just a map. It is a homomorphism modulo the ideal $I_k = (\pi_k^{v_k(p)-1}\delta_0)$. We can still define the space $D^a_{L/K}$ by just writing down equations of L/K and applying ψ termwise.

The key point of the whole proof is the amazing fact proved in [1, Theorem 7.2] that $D^a_{L/K}$ is finite ÉTALE over $A^1_K[0, \theta^a]$ if $a > b(L/K) - \epsilon$ for some $\epsilon > 0$, where $b(L/K)$ is the highest ramification break of L/K. This étaleness statement grants us the construction in Step II. The auxiliary étale locus given by ϵ enables us to read off spectral norms of differential equations from spectral norms of the base. This improvement will become important in Step IV.

The analogies of Step III and Step IV can be almost word by word copied from equal characteristic case.

Step V would be similarly trivial if ψ were a homomorphism. The failure of ψ being a homomorphism requires that we cannot change the definition equations for the extension too much under every operation of base change. For this, we have to "add a generic p-th roots of a p-basis" each time. We can also exhibit very delicate approximation to show that, after this base change, we can still get back to Abbes-Saito spaces via the $AS = TS$ Theorem.

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