Relative p-adic Hodge theory, II: (φ, Γ)-modules

Kiran S. Kedlaya and Ruochuan Liu

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Abstract

Building on foundations introduced in a previous paper, we give several p-adic analytic descriptions of the categories of étale $\mathbb{Z}_p$-local systems and étale $\mathbb{Q}_p$-local systems on an affinoid algebra over a finite extension of $\mathbb{Q}_p$ (or more generally, over the fraction field of the Witt vectors of a perfect field of characteristic p). These include generalizations of Fontaine's theory of (φ, Γ)-modules, the refinement of Fontaine's construction introduced by Cherbonnier and Colmez, and a recent description by Fargues and Fontaine in terms of semistable vector bundles on a certain scheme. Our descriptions depend on the embedding of the associated affinoid space into an affine toric variety, but there are natural functoriality maps relating the constructions for different choices of the embedding; these may be used to give analogous descriptions over more general rigid or Berkovich analytic spaces.

Contents

0 Introduction

After its formalization by Deligne [22], the subject of *Hodge theory* may be viewed as the study of the interrelationship among different cohomology theories associated to algebraic varieties over $\mathbb{C}$, most notably singular (Betti) cohomology and the cohomology of differential forms (de Rham cohomology). From work of Fontaine and others, there emerged a parallel subject of *p-adic Hodge theory* concerning the interrelationship among different cohomology theories associated to algebraic varieties over finite extensions of $\mathbb{Q}_p$, most notably étale cohomology with coefficients in $\mathbb{Q}_p$ and algebraic de Rham cohomology.

In ordinary Hodge theory, the relationship between Betti and de Rham cohomologies is forged using the *Riemann-Hilbert correspondence*, which relates topological data (local systems) to analytic data (integrable connections). In p-adic Hodge theory, one needs a similar correspondence relating de Rham data to étale $\mathbb{Q}_p$-local systems, which arise from the étale cohomology functor on schemes of finite type over K. However, in this case the local systems turn out to be far more plentiful, so it is helpful to first build a correspondence relating them to some sort of intermediate algebraic objects. This is achieved by the theory of *(φ, Γ)-modules*, which gives some Morita-type dualities relating étale $\mathbb{Q}_p$-local systems over K (i.e., continuous representations of the absolute Galois group G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces) with modules over certain mildly noncommutative topological algebras.

The latter appear as topological monoid algebras over commutative *period rings* for some continuous operators (the eponymous φ and Γ).

One of the key features of Hodge theory is that it provides information not just about individual varieties, but also about families of varieties through the mechanism of *variations of Hodge structures.* This paper is the second in a series beginning with [45], in which we develop some mechanisms for carrying out *relative p-adic Hodge theory*; in this paper in particular, we exhibit a generalization of the theory of (φ, Γ)-modules which describes the étale $\mathbb{Q}_p$-local systems on arbitrary nonarchimedean analytic spaces (in the sense of either Tate or Berkovich). In the remainder of this introduction, we recall the results of [45], then describe in detail what is accomplished in this paper and how the results of [45] are used. (We refer back to the introduction to [45] for additional background, including discussion of and contrast between two different possible relativizations of p-adic Hodge theory.)

0.1 φ-modules and local systems

For K a perfect field of characteristic p, the discrete representations of the absolute Galois group G_K of K on finite dimensional $\mathbb{F}_p$-vector spaces form a category equivalent to the category of *φ-modules* over K, i.e., finite-dimensional K-vector spaces equipped with isomorphisms with their φ-pullbacks. This amounts to a nonabelian generalization of the Artin-Schreier description of $(\mathbb{Z}/p\mathbb{Z})$-extensions of fields of characteristic p [23, Exposé XXII, Proposition 1.1]. Katz [40, Proposition 4.1.1] generalized this result in two directions, by exhibiting an equivalence of categories between φ-modules over the length n Witt vectors $W_n(R)$ over a perfect $\mathbb{F}_p$-algebra R and the locally constant sheaves in finite free $\mathbb{Z}/p^n\mathbb{Z}$-modules on the étale site of R. By taking inverse limits, one gets an equivalence of categories between φ-modules over the full p-typical Witt ring $W(R)$ over a perfect $\mathbb{F}_p$-algebra R and étale $\mathbb{Z}_p$-local systems on the small étale site of $\operatorname{Spec}(R)$. Similarly, one has an equivalence of categories between étale φ-modules over $W(R)[p^{-1}]$ and étale $\mathbb{Q}_p$-local systems on the small étale site of $\operatorname{Spec}(R)$.

The key result from [45] behind the existence of (φ, Γ)-modules and their generalizations is an equivalence of categories between the finite étale algebras over certain perfect $\mathbb{F}_p$-algebras and certain $\mathbb{Q}_p$-algebras (extending Faltings's *almost purity theorem*). For the purposes of this paper, we may state this result as follows. Let R be a perfect uniform Banach algebra over $\mathbb{F}_p((\overline{\pi}))$. Let $\mathfrak{o}_R$ be the subring of R consisting of elements of norm at most 1. Define the element $z = \sum_{i=0}^{p-1} [\overline{\pi} + 1]^{i/p} \in W(\mathfrak{o}_R)$. One then obtains an equivalence of categories between the finite étale algebras over R and over $W(\mathfrak{o}_R)[[\overline{\pi}]^{-1}]/(z)$ [45, Theorem 3.6.12].

Any rank-preserving equivalence of categories between the finite étale algebras over two different rings immediately gives corresponding equivalences of categories for $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems. One thus obtains an equivalence of categories between φ-modules over $W(R)[p^{-1}]$ and étale $\mathbb{Q}_p$-local systems on the small étale site of $\operatorname{Spec}(W(\mathfrak{o}_R)[[\overline{\pi}]^{-1}]/(z))$. One has some similar equivalences with $W(R)[p^{-1}]$ replaced with a certain subring of *overconvergent elements*, or a corresponding *extended Robba ring* giving by taking a certain Fréchet completion of the overconvergent subring. The last of these rings has the key property that it can be used to describe *analytic* étale $\mathbb{Q}_p$-local systems over $W(\mathfrak{o}_R)[[\overline{\pi}]^{-1}]/(z)$, which are obtained by glueing algebraic $\mathbb{Z}_p$-local systems up to isogeny not over a Zariski open covering, but rather over an open covering of the Gel'fand spectrum. One can also describe étale cohomology of the aforementioned local systems in terms of the corresponding φ-modules. See [45, §8] for all of these results.

0.2 (φ, Γ)-modules

Using the aforementioned results, one can describe étale $\mathbb{Q}_p$-local systems on some nonarchimedean analytic spaces over fields of mixed characteristic. Before indicating how this happens more generally, we first recall how one extracts the original theory of (φ, Γ)-modules for Galois representations over $\mathbb{Q}_p$. (A similar theory is available for finite extensions of $\mathbb{Q}_p$, or more generally finite extensions of an absolutely unramified p-adic field with perfect residue field.)

In the previous discussion, let us take $A = K = \mathbb{F}_p((\overline{\pi}))$; then the ring $F = W(\mathfrak{o}_R)[[\overline{\pi}]^{-1}]/(z)$ is none other than the p-adic completion of $\mathbb{Q}_p(\mu_{p^\infty})$ for the p-adic norm, with $[1+\overline{\pi}]^{p^{-n}}$ mapping to a primitive p^n-th root of unity for each nonnegative integer n. One can thus describe $\mathbb{Q}_p$-local systems over $\mathbb{Q}_p(\mu_{p^\infty})$ using φ-modules over $W(K)[p^{-1}]$, or over the overconvergent subring of same. To describe local systems over $\mathbb{Q}_p$, we must add some descent data for the action of the group $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$. This is done by lifting the group action to $W(\mathfrak{o}_R)$ so that $\gamma \in \mathbb{Z}_p^\times$ takes $[1+\overline{\pi}]$ to $[(1+\overline{\pi})^\gamma]$ (defined using the binomial series), and considering φ-modules equipped with isomorphisms with their γ-pullbacks, or (φ, Γ)-*modules*. We are particularly interested in the *étale* (φ, Γ)-modules; these satisfy a condition on the φ-action reminiscent of the semistability condition for vector bundles in geometric invariant theory. (We will return to this analogy later in this introduction.)

In the previous construction, it is relatively easy to pass from the ring $W(\mathfrak{o}_R)[p^{-1}]$ to the two-dimensional local field $\mathcal{E}$ defined as the completion of $\mathbb{Z}_p((\pi))[p^{-1}]$ for the p-adic norm. Since $\mathcal{E}$ has imperfect residue field, to make things unambiguous we must pick out a Frobenius lift on $\mathcal{E}$; we use the one sending $1+\pi$ to its p-th power. We then obtain a φ-equivariant embedding $\mathcal{E}$ into $W(\mathfrak{o}_R)[p^{-1}]$ taking $1+\pi$ to $[1+\overline{\pi}]$. The equivalence of étale (φ, Γ)-modules over $\mathcal{E}$ and over $W(\mathfrak{o}_R)[p^{-1}]$, as introduced by Fontaine [27], then follows from the fact that the field K and its completed perfection have equivalent categories of finite étale algebras.

A further refinement was achieved by Cherbonnier and Colmez [18], who showed that just as one can descend étale φ-modules over $W(\mathfrak{o}_R)[p^{-1}]$ to its overconvergent subring, one can descend étale (φ, Γ)-modules over $\mathcal{E}$ to its overconvergent subring $\mathcal{E}^\dagger$. This is not a purely formal observation, as it fails to hold for φ-modules. Rather, this is a subtler phenomenon similar in substance to Sen-Tate decompletion in Galois cohomology of local fields [64]. This refinement turns out to be extremely fruitful, as it allows for numerous constructions in p-adic Hodge theory to be made at the level of étale (φ, Γ)-modules. Many of these constructions pass through a further construction of Berger, extending scalars from $\mathcal{E}^\dagger$ to the *Robba ring* $\mathcal{R}$ of germs of analytic functions on open annuli of outer radius 1. (One formally defines the étale (φ, Γ)-modules over $\mathcal{R}$ to be the ones arising from base extension from $\mathcal{E}^\dagger$; it is easy to check that the base change functor from $\mathcal{E}^\dagger$ to $\mathcal{R}$ is an equivalence of categories.) For instance, Berger used this approach to give the first proof of Fontaine's conjecture that de Rham representations are potentially semistable [8], and to give a simple proof of the theorem of Colmez and Fontaine that weakly admissible filtered (φ, N)-modules are admissible [9]. (See the introduction to [45] for discussion of some other examples.)

0.3 Relative period rings

In this paper, we describe a generalization of the theory of (φ, Γ)-modules in which the field $\mathbb{Q}_p$ can be replaced by an arbitrary reduced affinoid algebra over $\mathbb{Q}_p$, or more generally, over $K = \operatorname{Frac} W(k)$ for k a perfect field of characteristic p. Note that this also includes affinoid algebras over finite extensions of K, by restriction of scalars. Before discussing the theory, we must indicate the base rings over which the relevant objects are defined.

As in the traditional theory, we cannot hope to directly realize local systems over our original affinoid algebra, but instead must pass to a highly ramified extension and then get back using descent. We do this by working not with a bare affinoid algebra, but rather with an affinoid algebra equipped with a certain choice of local coordinates. More precisely, we fix an unramified morphism from the corresponding reduced affinoid space to an affine space, or more generally to an affine toric variety. This time, in addition to adjoining the p-power roots of unity, we also adjoin the p-power roots of the coordinate functions. The completion of the result has the form $W(\mathfrak{o}_R)[[\overline{\pi}]^{-1}]/([1+\overline{\pi}])$ for a certain natural Banach algebra R over $k((\overline{\pi}))$.

From this construction, we make a suite of *period rings* equipped with an action of a Frobenius lift φ and an auxiliary group Γ_N of the form $\mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^n$, where n is the number of local coordinates. These are analogues of the rings of type $\mathbf{A}$ and $\mathbf{B}$ in the notation of Cherbonnier and Colmez [18], and of the *Robba ring* used in the work of Berger [8, 9]. (In the notation of this paper, the Robba ring is considered to be of a new type $\mathbf{C}$.) The setup is modeled closely on the original work of Faltings [25], but with the big difference that since we are not currently trying to study properties sensitive to integral models (like the crystalline condition), we are able to simplify and generalize by avoiding any use of integral models. (This avoidance of integral models is also a feature of the approaches to comparison isomorphisms described recently by Beilinson [7] and Scholze [63].)

In what follows, it will be important to distinguish between *perfect* and *imperfect* period rings; the distinction is approximately that φ is bijective on perfect period rings and not on imperfect ones. For instance, Fontaine's ring $\mathcal{E}$ from earlier is an example of an imperfect period ring; taking its weakly completed perfection with respect to the specified Frobenius lift φ gives a perfect period ring. One key distinction is that imperfect period rings are only defined when the map from the affinoid space to the affine toric variety is not just unramified but étale; in particular, this can only occur when the affinoid space admits at worst toroidal singularities. By contrast, perfect period rings are defined for unramified morphisms to toric varieties, which includes closed immersions; this is critical for discussion of functoriality, as indicated below.

0.4 Relative (φ, Γ)-modules

For the moment, let us continue to fix attention on a single affinoid space with a suitable choice of local coordinates. Using the results of [45], we obtain a correspondence between étale local systems (in $\mathbb{Z}_p$-modules or $\mathbb{Q}_p$-vector spaces) on the original affinoid algebra and certain modules over the period rings equipped with suitable actions of φ and Γ_N, called *étale* (φ, Γ_N)*-modules*. This includes an analogue of the decompletion argument of Cherbonnier and Colmez, which when suitably articulated carries through with little change. (The form of the argument we generalize is that given in [44], where the Tate-Sen formalism is avoided.)

Note that by working with type $\mathbf{C}$, we are able to describe *analytic* local systems on the original affinoid algebra, which in the case of a connected base ring correspond to representations of de Jong's étale fundamental group of the affinoid space [21]. These representations need not have compact image; for instance, Tate curves give rise to natural one-dimensional representations over $\mathbb{Q}_p$ with image $p^{\mathbb{Z}}$. (Further interesting examples arise on Rapoport-Zink period spaces; see below.) We also obtain descriptions of étale cohomology of local systems, generalizing results of Herr [35, 36] and Liu [53].

In addition, we give a relative analogue of a new variant of the theory of (φ, Γ)-modules recently introduced by Fargues and Fontaine [26]. In their work, one can replace the (φ, Γ)-modules over the Robba ring (including those not satisfying the étale condition) by an equivalent category of vector bundles over a certain scheme which are equivariant for an action of Γ on the base scheme. Note that φ is no longer present. The étaleness condition on (φ, Γ)-modules then becomes equivalent to *semistability* of vector bundles for a natural degree function.

0.5 Functoriality and globalization

As noted earlier, our description of relative (φ, Γ)-modules depends on a choice of local coordinates on an affinoid space. Since the category of étale $\mathbb{Q}_p$-local systems on that space does not distinguish any such choice, it is imperative to describe how changes of the choice of coordinates are reflected in the structure of relative (φ, Γ)-modules.

We handle this by considering the category of *torically framed affinoid algebras*, consisting of affinoid algebras equipped with unramified morphisms from the corresponding affinoid spaces to affine toric varieties, with morphisms induced by toric morphisms of the toric varieties. One recovers the usual category of affinoid algebras by formally inverting those morphisms which are split inclusions of toric varieties which act as the identity on the underlying affinoid spaces; we call these the *toric refinements*. We show that any toric refinement induces an equivalence of categories on (φ, Γ)-modules over the corresponding period rings of type $\tilde{\mathbf{C}}$. Note that we are limited to perfect period rings here because one cannot discuss toric refinements without using closed immersions, whereas imperfect period rings only behave well for étale maps to toric varieties.

This gives rise to a category of not necessarily étale (φ, Γ)-modules on a fairly general class of analytic spaces, by making a site of framed affinoid subdomains and considering crystals of (φ, Γ)-modules on these; among these, the étale objects again describe étale $\mathbb{Q}_p$-

local systems. An alternate approach has been introduced by Scholze [62] in the form of the *pro-étale site* of an analytic space; we will incorporate Scholze's point of view in a subsequent paper in this series. (Yet another approach would be to take the completed direct limit over *all* toric framings; this suggests an interpretation of p-adic Hodge theory in the language of Λ-rings.)

0.6 Further remarks

In this paper, the cyclotomic tower over $\mathbb{Q}_p$ plays a crucial role as a sufficiently ramified extension of $\mathbb{Q}_p$. (The key property is that the completion of the union over the tower is *perfectoid* in the sense of [45, Definition 3.5.1].) However, the setup of [45] leaves open the possibility of using other towers instead. For instance, based on Kisin's work on the classification of crystalline representations using Breuil modules [48], Caruso [17] has described (for p odd) an analogue of the theory of (φ, Γ)-modules where the role of the cyclotomic tower is played instead by the tower obtained by adjoining the p-power roots of p (or more generally of some uniformizer of a base field K). It seems likely that the optimal way to relate these constructions to ours and to extend them to the relative setting is to compare them to Scholze's canonically defined objects [62, 63].

One can employ relative (φ, Γ)-modules to construct some natural étale local systems arising in arithmetic geometry. In a subsequent paper, we plan to illustrate this by constructing natural local systems on Rapoport-Zink period domains of filtered isocrystals. This will realize the construction sketched in [43].

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1 Nonarchimedean analytic spaces

We continue the discussion of Berkovich analytic geometry initiated in [45]; besides some auxiliary lemmas, the main addition to [45, §2] is the introduction of analytic spaces over an analytic field. We retain all notation and terminology from [45, §2].

Hypothesis 1.0.1. Throughout this paper, fix a prime number p and an analytic field K. Until otherwise specified (see §3), K is permitted to carry the trivial norm, and its residue characteristic need not equal p.

1.1 Geometry of affinoid spaces

We need some assorted facts about affinoid spaces not included in [45, §2.4–2.5].

Convention 1.1.1. As in [45, Convention 2.4.2], we will make numerous statements in which the adverb *strictly* appears in parentheses. The intended interpretation of such statements is twofold: they hold when *strictly* is included at every instance (in case K has nontrivial norm), and they also hold when *strictly* is omitted at every instance.

Definition 1.1.2. For A an affinoid algebra over K, there exists a unique minimal finite subset S of $\mathcal{M}(A)$ with the property that for each $x \in A$, the maximum of $\alpha(x)$ over $\alpha \in \mathcal{M}(A)$ is achieved by some $\alpha \in S$ [12, Corollary 2.4.5]. The set S is called the *Shilov boundary* of A.

Lemma 1.1.3. *Let A be a (strictly) affinoid algebra over K. Let G be a finite group of bounded K-linear automorphisms of A.*

(a) The fixed subring A^G is closed in A and is (strictly) affinoid over K for the subspace norm.

(b) Via the map $\iota : A^G \to A$, A is a finite Banach algebra over A^G.

(c) The map $\iota^* : \mathcal{M}(A) \to \mathcal{M}(A^G)$ identifies $\mathcal{M}(A^G)$ with the quotient of $\mathcal{M}(A)$ by G in the category of topological spaces.

Proof. Parts (a) and (b) hold by [12, Proposition 2.1.14(ii)], so we concentrate on (c). The map $\iota^* : \mathcal{M}(A) \to \mathcal{M}(A^G)$ is surjective with finite fibres by [12, Corollary 2.1.16]. Since ι^* is a continuous surjection between compact spaces, it is automatically a quotient map [45, Remark 2.3.15(b)]; it thus remains to prove that G acts transitively on the fibres of ι^*.

Choose any $\alpha \in \mathcal{M}(A^G)$, and put $L = \mathcal{H}(\alpha)$, $A_L = A\widehat{\otimes}_K L$, and $A_L^G = (A_L)^G$. We may view A^G as the intersection of the kernels of the maps $g - 1$ for $g \in G$; these maps, being A^G-linear endomorphisms of the finite Banach A^G-module A, are necessarily strict by [45, Remark 2.5.3]. By [45, Lemma 2.2.10(b)], $A_L^G = (A^G)\widehat{\otimes}_K L$.

From the natural map $A^G \to \mathcal{H}(\alpha) = L$, we obtain a map $A_L^G = A^G\widehat{\otimes}_K L \to L\widehat{\otimes}_K L$; composing with the multiplication map $L\widehat{\otimes}_K L \to L$ gives a surjective map $A_L^G \to L$. Let $\mathfrak{p}$ be the kernel of this map; it is a maximal ideal of A_L^G. Note that G permutes the fibres of $\operatorname{Spec}(A_L) \to \operatorname{Spec}(A_L^G)$ transitively (see for instance [6, Exercise 5.13]), so G acts transitively on the set of primes of A_L above $\mathfrak{p}$.

Similarly, for each $\beta \in \mathcal{M}(A)$ with $\iota^*(\beta) = \alpha$, we get a map $A_L = A\widehat{\otimes}_K L \to \mathcal{H}(\beta)\widehat{\otimes}_K L \to \mathcal{H}(\beta)$ in which the last arrow is the multiplication map. The kernel of this map is a prime ideal $\mathfrak{q}$ lying above $\mathfrak{p}$, from which we may reconstruct β by extending the norm on the field $A_L^G/\mathfrak{p} \cong L$ to its finite extension $A_L/\mathfrak{q}$, then restricting along $A \to A_L \to A_L/\mathfrak{q}$. Consequently, G permutes the fibre of ι^* above α transitively, as desired. □

Theorem 1.1.4 (Gerritzen-Grauert, Temkin)**.** *A bounded homomorphism $A \to B$ in the category of (strictly) affinoid algebras over K is an epimorphism if and only if there exists* *a covering family $A \to B_1, \dots, A \to B_n$ of (strictly) rational localizations such that for each i, the composition $B_i \to B_i \widehat{\otimes}_A B$ can be written as a Weierstrass localization followed by a strict surjection.*

Proof. If $A \to B$ is an epimorphism, then the existence of the covering family follows in the strictly affinoid case from the Gerritzen-Grauert theorem, and in the general case by an extension due to Temkin; see [66, Theorem 3.1]. Conversely, suppose the covering family exists. Consider two bounded homomorphisms $B \to C$ of (strictly) affinoid algebras over K with the same restriction to A. For each i, we obtain two bounded homomorphisms $B_i \widehat{\otimes}_A B \to B_i \widehat{\otimes}_A C$ with the same restriction to B_i; these must coincide because $B_i \to B_i \widehat{\otimes}_A B$ has dense image and so must be an epimorphism. Hence the two compositions $B \to \oplus_i B_i \widehat{\otimes}_A B \to \oplus_i B_i \widehat{\otimes}_A C$ coincide. Since the maps $C \to B_i \widehat{\otimes}_A C$ form a covering family of C (see [45, Remark 2.4.5]), the map $C \to \oplus_i B_i \widehat{\otimes}_A C$ is injective by Tate's theorem [45, Theorem 2.5.13(a)]. It follows that the original maps $B \to C$ coincide, so $A \to B$ is an epimorphism. $\square$

Corollary 1.1.5. *Any affinoid subdomain of $\mathcal{M}(A)$ is a finite union of rational subdomains (but not conversely). If A is a strictly affinoid algebra over K, then any strictly affinoid subdomain of $\mathcal{M}(A)$ is a finite union of strictly rational subdomains (but not conversely).*

1.2 Analytic spaces

We now introduce analytic spaces, following [13]. The definition in [12] is somewhat more restrictive; see Remark 1.2.5. (One should ideally view Berkovich spaces in context together with Huber's *adic spaces* [38], but we omit the latter here.)

Definition 1.2.1. Let X be a topological space. Equip each subset of X with the induced topology. For τ a family of subsets of X and $Y \subseteq X$, put $\tau|_Y = \{V \in \tau : V \subseteq Y\}$. We say τ is a *patchwork* on X if it satisfies the following conditions.

(a) For each $V \in \tau$, $\tau|_V$ contains a fundamental system of neighborhoods in V around each point of V. (Recall that a neighborhood of x in V contains an open set in V containing x but need not itself be open in V.)

(b) For each $x \in X$, there exists a finite subfamily $\{V_1, \dots, V_n\} \subseteq \tau$ such that $x \in V_1 \cap \cdots \cap V_n$ and $V_1 \cup \cdots \cup V_n$ is a neighborhood of x in X.

(c) For each $U_1, U_2 \in \tau$ and each $x \in U_1 \cap U_2$, there exists a finite subfamily $\{V_1, \dots, V_n\} \subseteq \tau|_{U_1 \cap U_2}$ such that $x \in V_1 \cap \cdots \cap V_n$ and $V_1 \cup \cdots \cup V_n$ is a neighborhood of x in $U_1 \cap U_2$.

Example 1.2.2. An example of a patchwork on $\mathbb{R}$ is given by the collection of intersections $I \cap [m, n]$ for I an open interval and $m, n \in \mathbb{Z}$. More generally, let X be the total space of a locally finite simplicial complex; then the collection of all open subsets of all simplices in the complex forms a patchwork on X.

Remark 1.2.3. In [13], a family is called *dense* if it satisfies condition (a) of Definition 1.2.1, a *quasinet* if it satisfies conditions (a) and (b), and a *net* if it satisfies all three conditions. However, the term *net* has a far more established meaning in point-set topology, as a sequence indexed by a general directed set. (This concept is used to deal with spaces which are not first-countable; see for instance [46]. Such spaces can also be handled using *filters*, as in [15].) We have thus opted for a different name, which is meant to suggest a family of subsets which cover X, but only provides a neighborhood basis after the covering subsets are appropriately stitched together.

Definition 1.2.4. Let X be a locally Hausdorff topological space. A *(strictly) K-affinoid atlas* on X consists of the following data.

- (i) A patchwork τ on X.
- (ii) For each $U \in \tau$, a (strictly) K-affinoid algebra A_U and a homeomorphism $\mathcal{M}(A_U) \cong U$.
- (iii) For each $U, V \in \tau$ with $V \subseteq U$, a bounded homeomorphism $\mathrm{Res}_{U,V} : A_U \to A_V$ which identifies V with a (strictly) affinoid subdomain of U, satisfying the compatibility condition $\mathrm{Res}_{U,W} = \mathrm{Res}_{V,W} \circ \mathrm{Res}_{U,V}$.

Any (strictly) K-affinoid atlas can be uniquely extended to a maximal such atlas [13, Proposition 1.2.13]. We refer to a locally Hausdorff topological space equipped with a (strictly) K-affinoid atlas as a *(strictly) K-analytic space.* We form a category of (strictly) K-analytic spaces by starting with *strong morphisms* (continuous maps of spaces equipped with compatible morphisms of affinoids), then formally inverting those morphisms whose underlying maps are homeomorphisms; see [13, §1.2].

We will identify (strictly) K-affinoid spaces with certain (strictly) K-analytic spaces via the natural fully faithful (by Tate's theorem) functor. A *(strictly) affinoid subdomain* of a K-analytic space is an element of the maximal (strictly) K-affinoid atlas; this agrees with the previous definition for (strictly) K-affinoid spaces.

Remark 1.2.5. Note that the definition of a (strictly) K-analytic space does not guarantee that every point has an affinoid neighborhood (or equivalently a neighborhood basis consisting of affinoid spaces). Spaces with the latter property are said to be *good*; they may be viewed as locally ringed spaces for the usual topology on $\mathcal{M}(A)$ with local rings as described in [45, Definition 2.4.11]. Consequently, they coincide with the analytic spaces introduced in [12].

Remark 1.2.6. In Berkovich's original development, it was not checked that the natural functor from strictly K-analytic spaces to K-analytic spaces is fully faithful [13, Remark 1.5.6]. However, this was subsequently verified by Temkin [65].

Definition 1.2.7. Let A be a (strictly) K-affinoid algebra, and put $X = \mathcal{M}(A)$. The *weak G-topology* on X is the G-topology (set-theoretic Grothendieck topology) in which the admissible open sets are the (strictly) affinoid subdomains of X, and the admissible coverings are the finite coverings by (strictly) affinoid subdomains. For this G-topology, by the theorems of Tate and Kiehl [45, Theorem 2.5.13], the structure presheaf $\mathcal{O}_X : V \rightsquigarrow A_V$ is a sheaf of rings, over which any coherent module is represented by a finite A-module. By [45, Corollary 2.5.12], local freeness of a finite A-module can be checked locally in the G-topology.

The weak G-topology can be enriched in a natural way to provide a G-topology on any (strictly) K-analytic space X, in which the admissible open sets are the *(strictly) analytic subdomains* of X (those subsets on which the restriction of the maximal atlas forms a patchwork), and any admissible covering of a (strictly) affinoid subdomain can be refined to a finite covering by (strictly) affinoid subdomains; see [13, §1.3]. We use this G-topology to view X as a locally G-ringed space.

The relationship with Tate's theory of rigid analytic spaces is given by the following theorem.

Theorem 1.2.8. *The category of paracompact strictly K-analytic spaces is equivalent to the category of quasiseparated rigid K-analytic spaces which admit an admissible affinoid covering in which each set in the covering meets only finitely many others. In particular, the category of compact strictly K-analytic spaces is equivalent to the category of quasicompact quasiseparated rigid K-analytic spaces.*

Proof. See [13, Theorem 1.6.1]. □

Definition 1.2.9. Let A be an affinoid algebra over K, and let X be a scheme locally of finite type over $\operatorname{Spec}(A)$. Let $\mathcal{C}_A$ be the category of good K-analytic spaces Y equipped with morphisms to $\operatorname{Spec}(A)$ in the category of locally ringed spaces. Let $F : \mathcal{C}_A \to \mathbf{Set}$ be the functor taking $Y \in \mathcal{C}_A$ to the set of morphisms $Y \to \mathcal{M}(A)$ over $\operatorname{Spec}(A)$ in the category of locally ringed spaces. Then the functor F is representable by an object X^{an} [13, Proposition 2.6.1], called the *analytification* of X over A. (The case $A = K$ is treated in [12, §§3.4–3.5].)

Definition 1.2.10. Let $\psi : Y \to X$ be a morphism of analytic spaces over K. For $y \in Y$ an unspecified point, put $x = \psi(y)$. Let $\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}$ denote the local rings of X, Y at x, y, and let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$,

We say ψ is *finite* if for every affinoid subdomain U of X, $\psi^{-1}(U) \to U$ is induced by a finite morphism of affinoid algebras; this property is local on the target [13, Lemma 1.3.7]. We say ψ is *finite at y* if there exist open neighborhoods U, V of x, y such that the induced morphism $V \to U$ is open; one can then make these neighborhoods arbitrarily small [13, Lemma 3.1.2]. We say ψ is *quasifinite* if it is finite at each point of Y.

Assume for the rest of this definition that ψ is quasifinite. We say ψ is *flat at y* if there exist open neighborhoods U, V of x, y such that for any affinoid subdomain W of U, the induced map $\psi^{-1}(W) \to W$ is induced by a flat morphism of affinoid algebras. We say ψ is *flat* if it is flat at each point of Y.

We say ψ is *unramified at y* if the ring $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a finite separable field extension of $\mathcal{O}_{X,x}/\mathfrak{m}_x$. It is equivalent to require that the sheaf of relative differentials $\Omega_{Y/X}$ vanishes

[13, Corollary 3.3.6]. We say ψ is *unramified* if it is unramified at each point of Y. We say ψ is *étale* (at y or everywhere) if it is flat and unramified (at y or everywhere).

Remark 1.2.11. For good analytic spaces, any unramified morphism factors locally for the Berkovich topology on the source as a closed immersion followed by a finite étale morphism. Namely, by [13, Proposition 3.3.11] this reduces to the fact that an unramified (formally unramified and locally of finite type) morphism of schemes factors locally on the source as a closed immersion followed by an étale morphism [32, Corollaire 18.4.7]. For general analytic spaces, one gets a similar assertion working locally for the G-topology; it is unclear whether the same assertion holds for the Berkovich topology. One issue is the lack of canonicality of the factorization at the level of schemes, which can only be resolved at the level of algebraic spaces [60].

Similarly, using the Berkovich topology for good analytic spaces and the G-topology otherwise, any étale morphism factors locally on the source as a finite étale cover of an open immersion.

Remark 1.2.12. As for schemes (see [13, §4.1] for more details), we naturally associate to each K-analytic space an *étale site* and *étale topos*. These can be used to define étale fundamental groups of analytic spaces, as in [21]. For strictly K-analytic spaces, one can also obtain these objects in the framework of adic spaces [38].

1.3 Quasi-Stein spaces

Definition 1.3.1. An analytic space X over K is *quasi-Stein* if X can be written as an ascending sequence $U_0 \subseteq U_1 \subseteq \cdots$ of affinoid subdomains corresponding to a sequence $\cdots \to A_1 \to A_0$ of homomorphisms each with dense image. For example, any open disc is seen to be quasi-Stein by writing it as a union of closed discs with the same center. Similarly, any open or half-open annulus is quasi-Stein. (Note that by compactness, every affinoid subdomain of X is contained in some U_i.)

The following is a result of Kiehl [47, Satz 2.4] for strictly K-analytic spaces, but the proof applies to K-analytic spaces without change. Alternatively, one can reduce to Kiehl's results by extending the base field, as in [12, Chapter 2].

Theorem 1.3.2 (Kiehl). *Let X be a quasi-Stein space over K and let $\mathcal{F}$ be a coherent sheaf on X.*

(a) For every affinoid subdomain U of X, the map $\mathcal{F}(X) \to \mathcal{F}(U)$ has dense image.

(b) The sheaf $\mathcal{F}$ is acyclic for Čech cohomology.

(c) For every $\alpha \in X$, $\mathcal{F}(X)$ has dense image in the stalk $\mathcal{F}_\alpha$.

Corollary 1.3.3. *Let X be a quasi-Stein space over K and let $\mathcal{F}$ be a coherent sheaf on X. Let $s_1, \dots, s_n$ be global sections of $\mathcal{F}$ which generate $\mathcal{F}_\alpha$ for each $\alpha \in X$. Then $s_1, \dots, s_n$ generate $\mathcal{F}(X)$ as a module over $\mathcal{O}(X)$.*

Proof. Use $s_1, \dots, s_n$ to define a surjection $\mathcal{O}^n \to \mathcal{F}$, then apply Theorem 1.3.2 to the kernel. $\square$

Remark 1.3.4. For $f : X \to Y$ a finite surjective morphism of compact separated K-analytic spaces, the fact that X is affinoid does not imply that Y is affinoid, in contrast with the situation for schemes. This is related to the fact that there are compact separated K-analytic spaces which are not affinoid, but for which every coherent sheaf is acyclic. For instance, Qing Liu has exhibited examples which occur as immersed subspaces of a two-dimensional unit polydisc [51, 52].

2 Toric geometry

The methods of [45] can be used to describe étale local systems only on certain *deeply ramified* Banach rings (as in [30, §6] and [62]). To apply these results to affinoid algebras over a discretely valued field, it is helpful to have a systematic way to make deeply ramified covers. (An alternate approach would be to consider *all* deeply ramified covers; this is the point of view of Scholze. See Remark 3.9.7.)

Over the field itself, we may simply adjoin p-power roots of unity; however, for higher-dimensional spaces, one must also kill the differentials over the base field. Our approach to this issue is to consider not bare affinoid spaces, but affinoid spaces equipped with unramified maps to affine toric varieties; one may then use toric morphisms to add ramification in a natural way. This bears a strong resemblance to Payne's description of Berkovich analytification in terms of tropical geometry [58].

The reader unfamiliar with toric varieties may safely pretend on first reading that all affine toric varieties are affine spaces with marked coordinate axes. For additional background, see the survey article of Danilov [19], the introductory notes of Fulton [29], and the book of Oda [56].

Remark 2.0.1. Multiple conventions exist in the literature regarding the definition of the class of toric varieties. Under a definition in which toric varieties are taken to be partial torus compactifications, we will only be considering *normal* toric varieties.

2.1 Cones and fans

We begin with the underlying combinatorics behind toric varieties, and how it corresponds to geometry. (The reader unfamiliar with toric varieties is advised to read Example 2.1.5 first.)

Definition 2.1.1. A commutative monoid M (written multiplicatively) is *integral* if the cancellation law holds, i.e., if $ac = bc$ implies $a = b$. An equivalent condition is that M injects into its group completion M^{gp}. An integral monoid M is *saturated* if the image of M in M^{gp} is saturated, i.e., for all $x \in M^{\mathrm{gp}}$ such that $x^n \in M$ for some positive integer n, we have $x \in M$. We say M is *fine* if it is finitely generated and integral. We say M is *toric* if M is fine and saturated and M^{gp} is torsion-free.

Definition 2.1.2. Let N be a finite free $\mathbb{Z}$-module. A *convex polyhedral cone* in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a subset σ of the form $\{r_1\mathbf{v}_1 + \cdots + r_m\mathbf{v}_m : r_1, \dots, r_m \geq 0\}$ for some $\mathbf{v}_1, \dots, \mathbf{v}_m \in N_{\mathbb{R}}$; σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. The *dual cone* $\sigma^\vee$ consists of those functionals in $N^\vee_{\mathbb{R}} = N^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ which are nonnegative on σ; it is a convex polyhedral cone in $N^\vee_{\mathbb{R}}$ but is only strongly convex if σ has full dimension. (For instance, $\sigma = \{0\}$ is a strongly convex polyhedral cone with dual $\sigma^\vee = N^\vee_{\mathbb{R}}$.) We say σ is *rational* if $\mathbf{v}_1, \dots, \mathbf{v}_n$ can be taken in N, in which case $M_\sigma = \sigma^\vee \cap N^\vee$ is a toric monoid (Gordan's lemma; see [56, Proposition 1.1]). Conversely, for any toric monoid M, the $\mathbb{R}_{\geq 0}$-span of M is the dual cone of a strongly polyhedral strictly convex cone in $N_{\mathbb{R}}$ for $N = (M^{\mathrm{gp}})^\vee$.

A *face* of a cone σ is the intersection of σ with a supporting hyperplane. Note that if σ is a strongly convex rational polyhedral cone, then any face spans a rational subspace of $N_{\mathbb{R}}$, inside of which the face is itself a strongly convex rational polyhedral cone.

Definition 2.1.3. Let N be a finite free $\mathbb{Z}$-module. A *fan* in N is a finite set Δ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions.

(a) Each face of a cone in Δ is itself a cone in Δ.

(b) The intersection of two cones in Δ is a face of each.

Definition 2.1.4. Let R be any ring. An *affine toric variety* over R is a scheme of the form $\mathrm{Spec}(R[M_\sigma])$ for some strongly polyhedral rational polyhedral cone σ. (Here $R[M_\sigma]$ denotes the monoid ring over R.)

Let Δ be a fan in some finite free $\mathbb{Z}$-module N. For $\sigma, \tau \in \Delta$, the fan condition lets us identify $\mathrm{Spec}(R[M_{\sigma\cap\tau}])$ with an open affine subscheme of each of $\mathrm{Spec}(R[M_\sigma])$ and $\mathrm{Spec}(R[M_\tau])$. Using these identifications to glue the schemes $\mathrm{Spec}(R[M_\sigma])$ yields a scheme $X(\Delta)$, called the *toric variety* over R associated to Δ. Over a field, any toric variety is reduced, separated, and normal [29, §2.1] (but see Remark 2.0.1).

Example 2.1.5. Here are some basic examples of the previous construction, over an arbitrary ring R.

- For $\sigma = \mathbb{R}^n_{\geq 0}$, $R[M_\sigma] = R[T_1, \dots, T_n]$ is the ordinary polynomial ring, and $\mathrm{Spec}(R[M_\sigma])$ is the affine n-space over R.

- For $\sigma = \{0\} \subseteq \mathbb{R}^n$, $R[M_\sigma] = R[T_1^\pm, \dots, T_n^\pm]$ is the Laurent polynomial ring in n generators, and $\mathrm{Spec}(R[M_\sigma])$ is the n-dimensional (split) algebraic torus over R. Consequently, any toric variety may be viewed as a partial compactification of a torus.

- For n a positive integer, let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$, and put $\sigma_0 = \mathbb{R}^n_{\geq 0}$. For $i \in \{1, \dots, n\}$, let σ_i be the cone generated by $\mathbf{e}_j$ for $j \neq i$, together with $-\mathbf{e}_1 - \cdots - \mathbf{e}_n$. For Δ the fan $\{\sigma_0, \dots, \sigma_n\}$, the associated toric variety over R is the projective n-space over R.

Definition 2.1.6. Let Δ, Δ' be fans with ambient $\mathbb{Z}$-modules N, N'. A *morphism* $\Delta \to \Delta'$ of fans is a $\mathbb{Z}$-linear homomorphism $\varphi : N \to N'$ with the property that for $\varphi_{\mathbb{R}} : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ the $\mathbb{R}$-linear extension of φ, for each $\sigma \in \Delta$ there exists $\sigma' \in \Delta'$ for which $\varphi_{\mathbb{R}}(\sigma) \subseteq \sigma'$. Any such morphism induces a morphism of schemes $X(\Delta) \to X(\Delta')$ for any base ring; such a morphism of schemes is said to be *toric*. In particular, a morphism $\mathrm{Spec}(R[M_{\sigma'}]) \to \mathrm{Spec}(R[M_\sigma])$ of affine toric varieties is toric if and only if it is induced by a morphism of ambient $\mathbb{Z}$-modules whose $\mathbb{R}$-linear extension carries σ into σ'.

Since we will be studying analytic spaces by locally embedding them into toric varieties, it is natural to mention the following result about embedding schemes into toric varieties. (We do not know whether the same result holds over an arbitrary field.)

Theorem 2.1.7 (Włodarczyk)**.** *Let X be a normal reduced separated scheme of finite type over an algebraically closed field F. Then X admits a closed immersion into a toric variety over F if and only if any two closed points of X can be found inside some open affine subscheme of X. In this case, if X is also smooth, then it admits a closed immersion into a smooth toric variety.*

Proof. See [70, Theorem A]. □

It is also natural to point out the relationship with logarithmic structures. For more on these, see [39] or [57].

Definition 2.1.8. Let X be a scheme or a K-analytic space. A *prelogarithmic structure* (or *prelog structure*) on X is a sheaf of monoids M on the étale site of X equipped with a homomorphism $\alpha : M \to \mathcal{O}_X^*$ of sheaves of monoids, where $\mathcal{O}_X^*$ refers to the underlying multiplicative monoid of the structure sheaf. A prelogarithmic structure α is a *logarithmic structure* (or *log structure*) if α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$, where $\mathcal{O}_X^\times$ refers to the submonoid of $\mathcal{O}_X^*$ consisting of invertible sections. The forgetful functor from logarithmic to prelogarithmic structures has a left adjoint [39, §1.3].

A *chart* of a logarithmic structure α on X is a prelogarithmic structure of the form $\beta : P_X \to \mathcal{O}_X$ for P_X the constant sheaf defined by a fine monoid P together with an isomorphism of the associated logarithmic structure of β with α. A chart is *toric* if its underlying monoid P is toric. A logarithmic structure is *fine* (resp. *toric*) if it étale locally admits charts (resp. toric charts) everywhere.

Example 2.1.9. Any affine toric variety $\mathrm{Spec}(R[M_\sigma])$ admits a natural toric chart defined by the multiplicative map $M_\sigma \to R[M_\sigma]$. Consequently, any toric variety admits a natural logarithmic structure, and toric morphisms between toric varieties are precisely those that define morphisms of logarithmic structures.

2.2 Toric frames

We now use affine toric varieties to provide local coordinates on affinoid spaces.

Definition 2.2.1. For σ a strongly convex rational polyhedral cone, write $K(\sigma)$ for the analytification of $\mathrm{Spec}(K[M_\sigma])$. This space may be viewed as the union of the affinoid spaces $\mathcal{M}(K\{M_\sigma\}_\lambda)$ over all $\lambda \in N_\mathbb{R}$ (or all $\lambda \in N_\mathbb{Q}$), where $\mathcal{M}(K\{M_\sigma\}_\lambda)$ denotes the completion of $K[M_\sigma]$ for the weighted Gauss norm

$$\left| \sum_{s \in M_\sigma} c_s s \right|_\lambda = \max_{s \in M_\sigma} \{|c_s| p^{-\lambda(s)}\}.$$

To lighten notation, we identify elements of M_σ with functions on $K(\sigma)$ without putting brackets or other enclosing symbols around them.

By a *(strictly) rational subdomain* of $K(\sigma)$, we will mean a (strictly) rational subdomain U of $\mathcal{M}(K\{M_\sigma\}_\lambda)$ for some $\lambda \in N_\mathbb{Q}$. Note that U is then a (strictly) rational subdomain of $\mathcal{M}(K\{M_\sigma\}_\lambda)$ for any $\lambda \in N_\mathbb{Q}$ for which $U \subseteq \mathcal{M}(K\{M_\sigma\}_\lambda)$.

Definition 2.2.2. Let A be a reduced affinoid algebra over K. A *toric frame* for A is an unramified morphism $\psi : \mathcal{M}(A) \to K(\sigma)$. We say ψ is *boundary-free* if each $s \in M_\sigma$ maps to a unit in A, i.e., if ψ does not meet the boundary of the torus compactification $K(\sigma)$. In case $\sigma = \mathbb{R}^n_{\geq 0}$, we also call ψ an *affine frame*; the reader unfamiliar with toric varieties may safely restrict consideration to affine frames on first reading.

If $\mathcal{M}(B) \to \mathcal{M}(A)$ is an unramified morphism of affinoid spaces over K, then the composition $\psi' : \mathcal{M}(B) \to \mathcal{M}(A) \to K(\sigma)$ is again a toric frame. In case $\mathcal{M}(B)$ is a (strictly) affinoid, Weierstrass, Laurent, or rational subdomain of $\mathcal{M}(A)$, we say that ψ' is a *(strictly) affinoid, Weierstrass, Laurent*, or *rational subframe* of ψ. A finite collection of subframes $\psi_i : \mathcal{M}(B_i) \to K(\sigma)$ of ψ is a *covering family* (resp. a *strong covering family*) if $\mathcal{M}(A)$ is covered by the $\mathcal{M}(B_i)$ (resp. by the relative interiors of the $\mathcal{M}(B_i)$ in $\mathcal{M}(A)$).

Remark 2.2.3. Any affinoid algebra over K admits at least one toric frame, because any strict surjection $K\{M_\sigma\}_\lambda \to A$ defines such a frame.

Remark 2.2.4. A toric frame $\psi : \mathcal{M}(A) \to K(\sigma)$ gives rise naturally to logarithmic structures on $\mathrm{Spec}(A)$ and $\mathcal{M}(A)$, by pulling back the natural logarithmic structure on $K(\sigma)$.

Definition 2.2.5. We say a toric frame $\psi : \mathcal{M}(A) \to K(\sigma)$ is *étale* if the morphism ψ of analytic spaces is étale; if A is also strictly K-affinoid, we say that the frame ψ is *strictly étale*. We say that ψ is of *(strictly) rational type* if ψ factors as a finite étale cover of a (strictly) rational subdomain.

Remark 2.2.6. By Remark 1.2.11 plus Corollary 1.1.5, any (strictly) étale frame $\psi : \mathcal{M}(A) \to K(\sigma)$ admits a covering family of rational subframes of (strictly) rational type; however, the overlaps are not themselves guaranteed to be of rational type. The standard way to work around this is to construct a simplicial covering

$$\cdots \to \coprod_j \mathcal{M}(A_{2j}) \to \coprod_i \mathcal{M}(A_{1j}) \to \mathcal{M}(A)$$

of $\mathcal{M}(A)$ by rational subdomains in which each frame $\psi_{ij} : \mathcal{M}(A_{ij}) \to K(\sigma)$ is of (strictly) rational type.

It will occasionally be useful to replace a given toric frame by a framing of a slightly larger space without changing coordinates.

Definition 2.2.7. Let $\psi : \mathcal{M}(A) \to K(\sigma)$ be a toric frame. A *(strictly) étale enclosure* of ψ is a factorization $\mathcal{M}(A) \to \mathcal{M}(A') \to K(\sigma)$ of ψ in which $\mathcal{M}(A) \to \mathcal{M}(A')$ is a Runge immersion (a composition of a closed immersion with a Weierstrass subdomain embedding) and $\psi' : \mathcal{M}(A') \to K(\sigma)$ is a (strictly) étale frame. We will say also that ψ' is a (strictly) étale enclosure of ψ, with the choice of the factorization being understood.

Lemma 2.2.8. *Suppose that K is nontrivially normed. For any toric frame $\psi : \mathcal{M}(A) \to K(\sigma)$, the inverse limit of $\mathcal{M}(A')$ over all strictly étale enclosures $\psi' : \mathcal{M}(A') \to K(\sigma)$ is homeomorphic to $\mathcal{M}(A)$, and the completed direct limit of A' is isomorphic to A.*

Proof. Suppose first that ψ is a closed immersion. Pick any $\pi \in K$ with $0 < |\pi| < 1$. Choose $\lambda \in N_{\mathbb{Q}}$ for which ψ corresponds to a strict surjective homomorphism $\psi_\sharp : K\{M_\sigma\}_\lambda \to A$. Choose generators $b_1, \dots, b_m$ of the kernel of $\psi_\sharp$. For $l = 1, 2, \dots$, put

$$U_l = \{\alpha \in \mathcal{M}(K\{M_\sigma\}_\lambda) : \alpha(b_{1l}), \dots, \alpha(b_{ml}) \leq |\pi|^{-l}\};$$

these form a decreasing sequence of strictly Weierstrass subdomains of $\mathcal{M}(K\{M_\sigma\}_\lambda)$ with intersection $\psi(\mathcal{M}(A))$. This proves the first claim in this case; the second claim then follows from [45, Theorem 2.3.10].

In the general case, we may factor ψ through a closed immersion $\psi_0 : \mathcal{M}(A_0) \to K(\sigma)$ by taking A_0 to be the quotient of $K\{M_\sigma\}_\lambda$ by the kernel of $\psi_\sharp$. The map $\mathcal{M}(A) \to \mathcal{M}(A_0)$ is étale; for each $\alpha \in \mathcal{M}(A_0)$, we may use [45, Theorem 1.2.8, Lemma 2.4.12] to extend the finite étale morphism $\mathcal{M}(A) \times_{\mathcal{M}(A_0)} \mathcal{H}(\alpha) \to \mathcal{H}(\alpha)$ over some rational localization $A_0 \to B$ encircling α. These covers glue in a neighborhood of each point of $\mathcal{M}(A_0)$, and hence in a neighborhood of $\mathcal{M}(A_0)$. That neighborhood itself includes some strictly étale enclosure of ψ_0 by the previous paragraph, proving the claims. □

2.3 Maps between frames

So far, we have been working within a single toric variety. We next introduce some formalism for moving between different toric varieties.

Definition 2.3.1. For $\psi : \mathcal{M}(A) \to K(\sigma), \psi' : \mathcal{M}(A') \to K(\sigma')$ two toric frames, a *toric morphism* of frames from ψ' to ψ is a diagram

$$\begin{tikzcd} \mathcal{M}(A') \arrow[r, "\psi'"] \arrow[d] & K(\sigma') \arrow[d] \\ \mathcal{M}(A) \arrow[r, "\psi"] & K(\sigma) \end{tikzcd} \tag{2.3.1.1}$$

of morphisms of analytic spaces in which the right vertical arrow is toric. Such a morphism is a *toric refinement* if the map $\mathcal{M}(A') \to \mathcal{M}(A)$ is an isomorphism and the toric map $K(\sigma') \to K(\sigma)$ is induced by a splitting $M_{\sigma'} \cong M_\sigma \oplus T$ for some toric monoid T.

Definition 2.3.2. Let $\psi_1 : \mathcal{M}(A_1) \to K(\sigma_1)$ and $\psi_2 : \mathcal{M}(A_2) \to K(\sigma_2)$ be two toric frames, and let $\tau : \mathcal{M}(A_1) \to \mathcal{M}(A_2)$ be a morphism of K-affinoid spaces. We define the *framed graph* of τ to be the toric frame $\psi_3 : \mathcal{M}(A_3) \to K(\sigma_3)$ in which $A_3 = A_1$, $\sigma_3 = \sigma_1 \oplus \sigma_2$, and ψ_3 is obtained by identifying $K(\sigma_3)$ with $K(\sigma_1) \times_K K(\sigma_2)$ and taking the product of the morphisms ψ_1 and $\psi_2 \circ \tau$. From the projections out of the fibred product, we obtain toric morphisms $\psi_3 \to \psi_1$, $\psi_3 \to \psi_2$ whose underlying morphisms on affinoid spaces are the identification $\mathcal{M}(A_3) \cong \mathcal{M}(A_1)$ and the composition $\tau : \mathcal{M}(A_3) \cong \mathcal{M}(A_1) \to \mathcal{M}(A_2)$.

Example 2.3.3. In Definition 2.3.2, for τ an isomorphism, the framed graph is a toric refinement of both ψ_1 and ψ_2.

Remark 2.3.4. Using Remark 2.2.3 (to see that every affinoid space admits a frame) and Definition 2.3.2 (to lift every morphism of affinoid spaces), it follows that if we start with the category of toric frames, then localize by formally inverting the toric refinements, we recover the category of affinoid spaces over K. This will imply that when using toric frames to study affinoid spaces, most questions of functoriality will amount to checking compatibility of constructions of interest with the formation of toric refinements.

Remark 2.3.5. The analogue of Remark 2.3.4 would hold even if we used just locally closed immersions into toric varieties, rather than unramified morphisms. However, we will want to work with the étale topology, and it is easier to do so without having to change frames.

Remark 2.3.6. One can use toric frames to give a polyhedral interpretation of the geometry of nonarchimedean analytic spaces following Payne [58] or Hrushovski-Loeser [37]. Namely, for each strictly convex rational polyhedral cone σ, let $\tilde{\sigma}$ be the closure of σ in the space of linear maps $\sigma^\vee \to [0, +\infty]$. We then obtain a *tropicalization* map $\mathrm{Trop}(\sigma) : K(\sigma) \to \tilde{\sigma}$ characterized by

$$\mathrm{Trop}(\sigma)(\alpha)(s) = -\log(\alpha(s)) \qquad (\alpha \in K(\sigma), s \in M_\sigma).$$

For each frame $\psi : \mathcal{M}(A) \to K(\sigma)$, the composition $\mathrm{Trop}(\psi) : \mathcal{M}(A) \to K(\sigma) \to \tilde{\sigma}$ (which one might call the *tropicalization* of the frame ψ) has polyhedral image. The natural map from $\mathcal{M}(A)$ to the inverse limit of the images of the $\mathrm{Trop}(\psi)$ over all toric frames ψ is a homeomorphism (and likewise for affine frames). Moreover, a subset of $\mathcal{M}(A)$ is a rational subdomain if and only if it appears as the inverse image of a rational polyhedral set under some $\mathrm{Trop}(\psi)$.

Remark 2.3.7. Our use of framing data to study local systems on affinoid spaces admits parallels to several other constructions in algebraic geometry. Here are a few examples.

One can similarly work with framed affine schemes of finite type over a field, which come equipped with unramified morphisms to affine spaces. This gives an approach to the construction of algebraic de Rham cohomology (as in [34]): first define the cohomology of a framed affine scheme as the cohomology of the de Rham complex of its formal neighborhood in the ambient affine space, then show that toric refinements induce isomorphisms in cohomology.

A similar setting is the construction of rigid cohomology introduced by Berthelot. Given a framed affine scheme in characteristic p, one works with a certain "p-adic tubular neighborhood" of the scheme in the generic fibre of the formal affine space. See [50] for details. (In both cases, it is sometimes convenient to use more general smooth affine schemes as the ambient spaces, but affine spaces are sufficient to get the theory started.)

A different but loosely analogous construction is the *thickening space* introduced by Xiao [71, 72] in order to relate the Abbes-Saito ramification filtration on the Galois group of a local field with imperfect residue field [1, 2] to p-adic differential equations. It would be interesting to understand how these spaces relate to ours.

3 Perfect period rings

We now make perfect period rings associated to framed affinoid spaces over p-adic fields. The use of toric coordinates provides a link back to the results of [45] by letting us move from an affinoid space to a deeply ramified cover; this gives rise to a notion of relative (φ, Γ)-modules and a relationship between such objects and local systems. One also gets a relative form of the description of (φ, Γ)-modules in terms of vector bundles due to Fargues and Fontaine [26]. Finally, we globalize the construction by performing descent along toric refinements.

Our results extend work of many authors. To streamline the exposition, we have reserved most historical references to a separate section at the end of the paper (§4.10).

Hypothesis 3.0.1. For the remainder of the paper, put $K = \mathrm{Frac}(W(k))$ for k a perfect field of characteristic p. Put

$$K(\epsilon) = K[\mathbb{Q}_p/\mathbb{Z}_p] / \left(\sum_{i=0}^{p-1} [ip^{-1}] \right).$$

For $n \geq 0$, let ϵ_n be the class of $[p^{-n}]$ in $K(\epsilon)$; it is a primitive p^n-th root of unity. Fix a reduced affinoid algebra A over K and a toric frame $\psi : \mathcal{M}(A) \to K(\sigma)$ (Definition 2.2.2). Let α denote the spectral norm on A. Let N denote the ambient $\mathbb{Z}$-module of σ, put $M_\sigma = N^\vee \cap \sigma^\vee$, and put $N_p = N \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Notation 3.0.2. For d a positive integer, write $\mathbb{Z}_{p^d}$ for the finite étale $\mathbb{Z}_p$-algebra with residue ring $\mathbb{F}_{p^d}$, and write $\mathbb{Q}_{p^d}$ for $\mathbb{Z}_{p^d}[p^{-1}]$.

3.1 Toric frames and deeply ramified covers

In the usual theory of (φ, Γ)-modules, the first step in describing local systems over $\mathbb{Q}_p$ is to form the arithmetically profinite (and hence deeply ramified) extension $\mathbb{Q}_p(\epsilon)$. Using the toric frame ψ, we can make an analogous construction involving A. A similar construction has been introduced by Scholze in order to reduce certain cases of the weight-monodromy conjecture in étale cohomology from characteristic 0 to characteristic p [62].

Definition 3.1.1. For $n = 0, 1, \dots$, let $\varphi_n : K(\epsilon_{n+1})(\sigma) \to K(\epsilon_n)(\sigma)$ be the morphism acting as the Witt vector Frobenius on K, embedding $K(\epsilon_n)$ into $K(\epsilon_{n+1})$ within $K(\epsilon)$, and taking $s \in M_\sigma$ to s^p. Put $\Phi_n = \varphi_0 \circ \cdots \circ \varphi_{n-1}$. Put

$$\mathcal{M}(A_{\psi,n}) = \mathcal{M}(A) \times_{K(\sigma),\Phi_n} K(\epsilon_n)(\sigma)$$

in the category of reduced analytic spaces; that is, $A_{\psi,n}$ is the reduced quotient of $A \otimes_{K[S_\sigma],\Phi_{n,\sharp}} K(\epsilon_n)[S_\sigma]$. From this description, $A_{\psi,n+1}$ is a finite $A_{\psi,n}$-module, and (since $A_{\psi,n}$ is reduced) the induced map $\varphi_{n,\sharp} : A_{\psi,n} \to A_{\psi,n+1}$ is injective. Equip $A_{\psi,n}$ with its spectral norm $\alpha_{\psi,n}$; then $\varphi_n : \mathcal{M}(A_{\psi,n+1}) \to \mathcal{M}(A_{\psi,n})$ is surjective by [12, Corollary 2.1.16]. By interpreting $\alpha_{\psi,n}$ as a supremum norm [45, Theorem 2.3.10], we deduce that $\varphi_{n,\sharp}$ is isometric. Let $A_{\psi,\infty}$ be the completed direct limit of the $A_{\psi,n}$, and let $\alpha_{\psi,\infty}$ be the induced power-multiplicative norm on $A_{\psi,\infty}$.

Remark 3.1.2. If ψ is boundary-free, then the morphisms $A_{\psi,n} \to A_{\psi,n+1}$ are étale. Otherwise, they are only log-étale for the toric logarithmic structures indicated in Remark 2.2.4. This will force to keep track of certain extra conditions when trying to perform Galois descent; see for example Definition 3.3.5.

Note that the tensor product $A \otimes_{K[S_\sigma],\Phi_{n,\sharp}} K(\epsilon_n)[S_\sigma]$ is itself reduced if ψ is boundary-free (because an étale algebra over a reduced ring is étale). The same also holds if the image of each irreducible component of $\mathcal{M}(A)$ under ψ meets the torus inside $K(\sigma)$, as we may reduce to the boundary-free case using Lemma 2.2.8.

Remark 3.1.3. For $\psi' : \mathcal{M}(A') \to K(\sigma)$ running over all strictly étale enclosures of ψ and n running over all nonnegative integers, the base change functor $\mathbf{F\acute{E}t}(\varinjlim A'_{\psi',n}) \to \mathbf{F\acute{E}t}(A_{\psi,\infty})$ is a tensor equivalence by Lemma 2.2.8 plus [45, Proposition 2.6.9].

Definition 3.1.4. Let $\mathfrak{o}_{\overline{A}_\psi}$ be the inverse perfection of $\mathfrak{o}_{A_{\psi,\infty}}$ [45, Definition 3.4.1]. For $\theta : W(\mathfrak{o}_{\overline{A}_\psi}) \to \mathfrak{o}_{A_{\psi,\infty}}$ the map defined as in [45, Definition 3.4.3], we may equip $\mathfrak{o}_{\overline{A}_\psi}$ with the power-multiplicative seminorm $\overline{\alpha}_\psi = \mu(\theta^*(\alpha_{\psi,\infty}))$. By [45, Lemma 3.4.5], $\overline{\alpha}_\psi$ is a norm under which $\mathfrak{o}_{\overline{A}_\psi}$ is complete. Put $\omega = p^{-p/(p-1)}$, $\overline{\pi} = (\dots, \epsilon_1 - 1, \epsilon_0 - 1) \in \mathfrak{o}_{\overline{A}_\psi}$, and $\pi = [1 + \overline{\pi}] - 1 \in W(\mathfrak{o}_{\overline{A}_\psi})$. Note that

$$\overline{\alpha}_\psi(x\overline{\pi}) = \omega\overline{\alpha}_\psi(x) \qquad (x \in \mathfrak{o}_{\overline{A}_\psi}). \tag{3.1.4.1}$$

We extend $\overline{\alpha}_\psi$ multiplicatively to $\overline{A}_\psi = \mathfrak{o}_{\overline{A}_\psi}[\overline{\pi}^{-1}]$ so that (3.1.4.1) holds also for $x \in \overline{A}_\psi$. The map θ extends to a map $\theta : \tilde{\mathcal{R}}^{\mathrm{int},1}_{\overline{A}_\psi} \to A_{\psi,\infty}$; we will use the symbol θ to refer to the extended map unless otherwise specified.

Example 3.1.5. Suppose that $A = K$. By [45, Example 3.3.8], θ is surjective and $\overline{A}_\psi$ is the completed perfection of $k((\overline{\pi}))$.

Example 3.1.6. Suppose that $A = K\{M_\sigma\}_\lambda$ for some $\lambda \in N_{\mathbb{Q}}$. In this case, the fact that $\overline{\varphi}$ is surjective on $\mathfrak{o}_{A_{\psi,\infty}}/(p)$ follows from the corresponding fact for $A = K$ (Example 3.1.5). We may identify $\overline{A}_\psi$ with the completed direct perfection R_λ of $k((\overline{\pi}))\{M_\sigma\}_\lambda$.

Theorem 3.1.7. *(a) The maps* $\theta : \tilde{\mathcal{R}}^{\mathrm{int},1}_{\overline{A}_\psi} \to A_{\psi,\infty}$ *and* $\overline{\varphi} : \mathfrak{o}_{A_{\psi,\infty}}/(p) \to \mathfrak{o}_{A_{\psi,\infty}}/(p)$ *are surjective.*

(b) For every $x \in A_{\psi,\infty}$*, there exists* $y = \sum_{i=0}^{\infty} [\overline{y}_i] p^i \in W(\mathfrak{o}_{\overline{A}_\psi})[[\overline{\pi}]^{-1}]$ *with* $\theta(y) = x$ *and* $\overline{\alpha}_\psi(y_0) \geq \overline{\alpha}_\psi(y_i)$ *for all* $i > 0$*. In particular,* $\lambda(\overline{\alpha}_\psi)(y) = \alpha_{\psi,\infty}(x)$*, so* θ *is optimal (and hence strict).*

(c) The kernel of θ *is generated by* $z = \sum_{i=0}^{p-1} [\overline{\pi} + 1]^{i/p}$*.*

(d) There is a natural homeomorphism $\mathcal{M}(A_{\psi,\infty}) \cong \mathcal{M}(\overline{A}_\psi)$ *under which (strictly) rational subdomains on both sides correspond. Moreover, if* $\overline{B}$ *represents a rational subdomain of* $\mathcal{M}(\overline{A}_\psi)$*, then* $W(\mathfrak{o}_{\overline{B}})[[\overline{\pi}]^{-1}]/(z)$ *represents the corresponding rational subdomain of* $\mathcal{M}(A_{\psi,\infty})$*.*

(e) The Banach $\mathbb{Q}_p$*-algebra* $A_{\psi,\infty}$ *is perfectoid in the sense of [45, Definition 3.6.1].*

(f) There is an equivalence of tensor categories between $\mathbf{F\acute{E}t}(A_{\psi,\infty})$ *and* $\mathbf{F\acute{E}t}(\overline{A}_\psi)$ *under which* $\overline{B} \in \mathbf{F\acute{E}t}(\overline{A}_\psi)$ *corresponds to* $W(\mathfrak{o}_{\overline{B}})[[\overline{\pi}]^{-1}]/(z)$*. Moreover, the objects of both categories may be naturally viewed as uniform Banach algebras which are finite Banach algebras over their respective base rings.*

Proof. We start with a series of observations.

(i) By [45, Example 3.3.5], the element z described in (c) is primitive of degree 1 in the sense of [45, Definition 3.3.4].

(ii) By (i), if θ has dense image, then (a) holds by [45, Lemma 3.4.8]. Conversely, surjectivity of $\overline{\varphi}$ implies surjectivity of θ, in which case (a) also holds.

(iii) Given (a), we may deduce (b) and (c) from [45, Lemma 5.5.5], (d) from [45, Theorem 3.3.7, Theorem 3.6.8], and (e) and (f) from [45, Theorem 3.6.12].

By these observations plus Example 3.1.6, the claim holds when $A = K\{M_\sigma\}_\lambda$ for some $\lambda \in N_{\mathbb{Q}}$. Feeding (d)-(f) back into (i)-(iii), we deduce the claim in case ψ is of rational type.

Suppose next that ψ is étale; we treat this case using a suggestion of Peter Scholze (noting that a treatment using almost commutative algebra would be somewhat briefer). Form a simplicial covering of $\mathcal{M}(A)$ as in Remark 2.2.6. For each n, we have a strict exact sequence

$$0 \to A_{\psi,n} \to \bigoplus_j A_{\psi_{1j},n} \to \bigoplus_j A_{\psi_{2j},n} \to \cdots .$$

Take direct limits to obtain another exact sequence; since everything is known for each ψ_{ij}, we may complete each term except the first to obtain an exact sequence in which each arrow is almost optimal [45, Proposition 3.6.9]. Pick any $\lambda \in K(\epsilon)$ with $p^{-1/p^2} < |\lambda| < 1$. For each $x \in \cup_n \mathfrak{o}_{A_{\psi,n}}$, we can find $y \in \oplus_j \mathfrak{o}_{A_{\psi_{1j},\infty}}$ for which $y^p - x$ is divisible by p in $\oplus_j \mathfrak{o}_{A_{\psi_{1j},\infty}}$ and hence in $\oplus_j \mathfrak{o}_{A_{\psi_{2j},\infty}}$. In the latter ring, x vanishes, so y^p is divisible by p and hence by

λ^{p^2}, and y is divisible by λ^p. By almost optimality (i.e., the coincidence of subspace and quotient norms), it follows that in $\oplus_j \mathfrak{o}_{A_{\psi_{1j},\infty}}$ we can find some z congruent to y modulo λ and mapping to zero in $\oplus_j \mathfrak{o}_{A_{\psi_{2j},\infty}}$. By viewing z as an element of $A_{\psi,\infty}$ (and recalling that $\cup_n A_{\psi,n}$ is dense in $A_{\psi,\infty}$), we conclude that $\overline{\varphi}$ is surjective on $\mathfrak{o}_{A_{\psi,\infty}}/(\lambda)$ and hence on $\mathfrak{o}_{A_{\psi,\infty}}/(p)$ by [45, Lemma 3.4.2]. By (i)-(iii), we may deduce (a)-(e) for A.

To deduce the general case, apply Lemma 2.2.8 to produce a strictly étale enclosure $\psi' : \mathcal{M}(B) \to K(\sigma)$ of ψ. For each positive integer n, $B_{\psi',n} \to A_{\psi,n}$ has dense image, as then does $B_{\psi',\infty} \to A_{\psi,\infty}$. By this observation plus the previous case, the composition

$$\tilde{\mathcal{R}}^{\mathrm{int},1}_{\overline{B}_{\psi'}} \xrightarrow{\theta} B_{\psi',\infty} \to A_{\psi,\infty}$$

has dense image. Since this map factors through $\theta : \tilde{\mathcal{R}}^{\mathrm{int},1}_{\overline{A}_{\psi}} \to A_{\psi,\infty}$, the latter also has dense image. By (i)-(iii), the theorem holds in this case. □

We also obtain a generalization of Faltings's almost purity theorem [24, 25]. We will not use this result here, but it may be relevant for some other applications. For almost ring theory terminology used here, see [45, Definition 5.5.8] for just the definitions, or [30] for full context.

Theorem 3.1.8. *For $B \in \mathbf{F\acute{E}t}(A)$, $\cup_n \mathfrak{o}_{A_{\psi,n}\otimes_A B}$ is uniformly almost finite projective and almost finite étale over $\cup_n \mathfrak{o}_{A_{\psi,n}}$.*

Proof. This follows from the corresponding statement after completion, which is a case of [45, Theorem 5.5.9]. □

Remark 3.1.9. Essentially the same result as Theorem 3.1.8 has been obtained by Scholze [63]; the two arguments are distinct, but resemble each other more than either resembles the original method of Faltings. See [45, Remark 5.5.10] for more discussion.

It will be convenient to have a more precise description of $\overline{A}_\psi$ in case ψ is a rational subdomain.

Lemma 3.1.10. *Suppose that ψ is a (strictly) rational subdomain, and choose $\lambda \in N_{\mathbb{Q}}$ so that $\psi_\sharp$ factors through $K\{M_\sigma\}_\lambda$. Then the induced map $R_\lambda \to \overline{A}_\psi$ is a (strictly) rational localization defined by some $\overline{f}_1, \dots, \overline{f}_m, \overline{g} \in k[\overline{\pi}][S_\sigma]$ and some real numbers $p_1, \dots, p_m > 0$, with $p_1 = \cdots = p_m = 1$ in the strictly rational case.*

Proof. By applying Theorem 3.1.7, we obtain $\overline{f}_1, \dots, \overline{f}_m, \overline{g} \in R_\lambda$ and $p_1, \dots, p_m > 0$ defining a rational localization $R_\lambda \to \overline{A}_\psi$, with $p_1 = \cdots = p_m = 1$ in the strictly rational case. It remains to force the ring elements into the desired subring. By [45, Remark 2.4.8], we can also take $\overline{f}_1, \dots, \overline{f}_m, \overline{g}$ in the dense subring $\cup_{i=0}^\infty \overline{\varphi}^{-i}(k[\overline{\pi}^{\pm}][M_\sigma])$ of R_λ. By applying $\overline{\varphi}$ repeatedly, we obtain $\overline{f}_1, \dots, \overline{f}_m, \overline{g} \in k[\overline{\pi}^{\pm}][M_\sigma]$ of the desired form. By multiplying through by a power of $\overline{\pi}$, we may force $\overline{f}_1, \dots, \overline{f}_m, \overline{g} \in k[\overline{\pi}][M_\sigma]$ as desired. □

Corollary 3.1.11. *Suppose that the frame ψ is of (strictly) rational type. Then $\overline{A}_\psi$ is the completed perfection of a (strictly) affinoid algebra over $k((\overline{\pi}))$.*

Proof. Combine Lemma 3.1.10 with Theorem 3.1.7(d). □

Remark 3.1.12. Corollary 3.1.11 can fail when ψ is not of rational type, because the ring $\overline{A}_\psi$ can have infinitely many idempotent elements. Perhaps the simplest example is when $K(\sigma)$ is the one-dimensional affine space over K with coordinate T and $A = K\{T\}/(T-1)$.

3.2 Action of Γ

The passage from A to $A_{\psi,\infty}$ gives rise to a Galois group which we must keep track of as we go along.

Definition 3.2.1. Let Γ_N be the semidirect product $\mathbb{Z}_p^\times \ltimes N_p$. Define an action of Γ_N on $A_{\psi,n}$ via isometric automorphisms, with $\gamma \in \mathbb{Z}_p^\times$ acting via the cyclotomic character on $K(\epsilon_n)$ (i.e., carrying ϵ_n to ϵ_n^γ) and fixing M_σ, and $\nu \in N_p$ acting via the formula

$$\nu\left(\sum_{s\in M_\sigma} c_s s\right) = \sum_{s \in M_\sigma} \epsilon_n^{\langle \nu, s\rangle} c_s s.$$

A more compact way to express this action is to express any $\gamma \in \Gamma_N$ as $\nu \cdot \gamma_0$ with $\nu \in N_p, \gamma_0 \in \mathbb{Z}_p^\times$ and set $\langle \gamma, s\rangle = \langle \nu, s\rangle$, so that $\gamma(s) = \epsilon_n^{\langle \gamma, s\rangle} s$ for all $\gamma \in \Gamma_N$. The maps φ_n are equivariant for these actions, so we get an action on $A_{\psi,\infty}$; this action is continuous because Γ_N acts by isometries and each element of the dense subring $\cup_n A_{\psi,n}$ of $A_{\psi,\infty}$ is fixed by some open subgroup of Γ_N. This action transfers to a continuous action on $\overline{A}_\psi$ with the property that

$$\gamma(1+\overline{\pi}) = (1+\overline{\pi})^{\gamma_0}, \qquad \gamma(s) = (1+\overline{\pi})^{\langle \nu, s\rangle} s \qquad (\gamma = \nu\cdot\gamma_0, \nu \in N_p, \gamma_0 \in \mathbb{Z}_p^\times).$$

We may identify $\mathcal{M}(\overline{A}_\psi) \cong \mathcal{M}(A_{\psi,\infty})$ (see Theorem 3.1.7) with the inverse limit of the $\mathcal{M}(A_{\psi,n})$. By applying Lemma 1.1.3 to each $\mathcal{M}(A_{\psi,n})$, we deduce that the projection $\mathcal{M}(A_{\psi,\infty}) \to \mathcal{M}(A)$ identifies $\mathcal{M}(A)$ with the quotient of $\mathcal{M}(A_{\psi,\infty}) \cong \mathcal{M}(\overline{A}_\psi)$ by the action of Γ_N.

Remark 3.2.2. Note that if $\epsilon < 1$ and $x \neq g(x)$ for some $x \in \overline{A}_\psi$ and some $g \in U$, then there exists a nonnegative integer n for which

$$\overline{\alpha}_\psi(x^{p^{-n}} - g(x^{p^{-n}})) > \epsilon\overline{\alpha}_\psi(x^{p^{-n}}).$$

It follows that while the actions of Γ_N on $\overline{A}_\psi$ and $A_{\psi,\infty}$ are continuous, they are not in general analytic; that is, one does not obtain actions of the Lie algebra of Γ_N. This can only be remedied by restricting ψ suitably and then forming imperfect period rings; see §4.3.

Lemma 3.2.3. *For $R \in$ **FÉt**$(\overline{A}_\psi)$ and $B \in$ **FÉt**$(A_{\psi,\infty})$ corresponding via Theorem 3.1.7(f) and admitting compatible actions of Γ_N, the following conditions are equivalent.*

(a) The action of Γ_N on R is continuous.

(b) The action of Γ_N on B is continuous.

(c) There exist a positive integer n, an object $B_n \in \mathbf{F\acute{E}t}(A_{\psi,n})$, an action of Γ_N on B_n which is trivial on some open subgroup of Γ_N, and a Γ_N-equivariant isomorphism $B_n \otimes_{A_{\psi,n}} A_{\psi,\infty} \cong B$.

Proof. The equivalence of (a) and (b) follows by writing

$$B = W(\mathfrak{o}_R)[[\overline{\pi}]^{-1}]/(z), \qquad R = \mathfrak{o}_B^{\mathrm{frep}}[\overline{\pi}^{-1}].$$

It is clear that (c) implies (b), so we need only check that (b) implies (c); moreover, by [45, Theorem 2.6.10], we may work locally around some $\beta \in \mathcal{M}(A)$.

Suppose that the action of Γ_N on B is continuous. From Remark 3.1.3, we obtain a positive integer m, an element $B_m \in \mathbf{F\acute{E}t}(A_{\psi,m})$, and an isomorphism $B_m \otimes_{A_{\psi,m}} A_{\psi,\infty} \cong B$. Pick any $\beta_m \in \mathcal{M}(A_{\psi,m})$ lifting β. By passing from A to a suitable rational localization encircling β, we may ensure that $B_m = A_{\psi,m}[T]/(P(T))$ for some $P(T) \in A_{\psi,m}[T]$ such that for $x \in B_m$ the image of T, $P'(x)$ vanishes nowhere on $\mathcal{M}(B_m)$ and hence is a unit in B_m by [45, Corollary 2.3.7].

Write $Q(T) = P(T+x) = \sum_{i>0} Q_i T^i$ with $Q_i \in B_m$ and $Q_1 \in B_m^\times$. For γ in any open subgroup of Γ_N fixing $A_{\psi,m}$, we have $\gamma(P) = P$, so $0 = \gamma(P(x)) = \gamma(P)(\gamma(x)) = P(\gamma(x)) = Q(\gamma(x) - x)$. However, because Γ_N acts continuously on B, for γ in a sufficiently small open subgroup of Γ_N, we have

$$\alpha_{\psi,\infty}(Q(\gamma(x) - x)) = \alpha_{\psi,\infty}(Q_1(\gamma(x) - x)) \geq \alpha_{\psi,\infty}(Q_1^{-1})^{-1} \alpha_{\psi,\infty}(\gamma(x) - x).$$

It follows that γ fixes x, from which the desired result follows. □

Remark 3.2.4. When ψ is étale, the equivalent conditions of Lemma 3.2.3 are always satisfied. See Proposition 3.5.7 for an even stronger statement.

3.3 Local systems

Using Γ, we can relate local systems over A, $A_{\psi,\infty}$, and $\overline{A}_\psi$ as follows. For convenience, we begin by recopying some relevant facts about faithfully flat descent from [45, Theorems 1.3.4 and 1.3.5].

Theorem 3.3.1 (Faithfully flat descent)**.** *Let $f : R \to S$ be a faithfully flat morphism of rings.*

(a) The morphism f is an effective descent morphism for the category of modules over rings.

(b) An R-module U is finite (resp. finite projective) if and only if $f^*U = U \otimes_R S$ is a finite (resp. finite projective) S-module.

(c) An R-algebra U is finite étale if and only if f^*U is a finite étale S-algebra.

Definition 3.3.2. Let G be a group acting on a ring R. Let c, d be integers with $d > 0$. Let E be an étale sheaf, étale $\mathbb{Z}_{p^d}$-local system, étale $\mathbb{Q}_{p^d}$-local system, or étale (c, d)-$\mathbb{Q}_p$-local system on $\mathrm{Spec}(R)$. An *action* of G on E is given by specifying for each $g \in G$ an isomorphism $\iota(g) : E \cong g^*E$, subject to the restriction that $\iota(g_1 g_2) = g_2^*(\iota(g_1)) \circ \iota(g_2)$. For example, if G acts trivially on R, there is a *trivial action* on E using the natural identifications $E \cong g^*E$ for all $g \in G$.

Definition 3.3.3. Let G be a profinite topological group acting continuously on a Banach ring R. Suppose first that E is a locally constant $\mathbb{Z}/p^m\mathbb{Z}$-sheaf on the étale site of $\mathrm{Spec}(R)$ on which G acts. The sheaf E is then represented by a finite étale R-algebra R_m on which G also acts. Equip R_m with a topology by viewing it as a finite projective R-algebra and arguing as in [45, Lemma 2.2.12]. We say that the action of G on E is *continuous* if the action of G on R_m is continuous.

Suppose next that $E = \{\cdots \to E_2 \to E_1\}$ is an étale $\mathbb{Z}_{p^d}$-local system on $\mathrm{Spec}(R)$. We say that an action of G on E is *continuous* if the action of G on E_m is continuous for each m.

Suppose next that E is an étale $\mathbb{Q}_{p^d}$-local system on $\mathrm{Spec}(R)$. Write $E = F \otimes_{\mathbb{Z}_{p^d}} \mathbb{Q}_{p^d}$ for some $\mathbb{Z}_{p^d}$-local system F on $\mathrm{Spec}(R)$, on which G need not act. We say that an action of G on E is *continuous* if for some (and hence any) choice of F, there is an open subgroup H of G which acts continuously on F.

Suppose next that E is an étale (c, d)-$\mathbb{Q}_p$-local system on $\mathrm{Spec}(R)$. We say that an action of G on E is *continuous* if the action on the underlying $\mathbb{Q}_{p^d}$-local system is continuous.

Suppose finally that E is an étale (c, d)-$\mathbb{Q}_p$-local system on $\mathcal{M}(R)$. We say that an action of G on E is *continuous* if there exist an open subgroup H of G and a covering family $R \to R_1, \ldots, R \to R_n$ of rational localizations stable under H, such that E restricts to an étale (c, d)-$\mathbb{Q}_p$-local system on $\mathrm{Spec}(R_1 \oplus \cdots \oplus R_n)$ on which H acts continuously.

Remark 3.3.4. In the case $R \in \mathbf{F\acute{E}t}(A_{\psi,\infty})$ and $G = \Gamma_N$, Lemma 3.2.3 implies that an action of G on an étale $\mathbb{Z}_{p^d}$-local system $E = \{\cdots \to E_2 \to E_1\}$ is continuous if and only if for each positive integer m, there exist a nonnegative integer n and an open subgroup H of G fixing $A_{\psi,n}$ such that the action of H on E_m arises from the trivial action on some locally constant $\mathbb{Z}/p^m\mathbb{Z}$-sheaf on $\mathrm{Spec}(A_{\psi,n})$. Using this interpretation, the continuity condition may be seen to be local for the étale topology.

In the case at hand, we need an additional condition besides continuity.

Definition 3.3.5. Let E be an étale sheaf, étale $\mathbb{Z}_{p^d}$-local system, étale $\mathbb{Q}_{p^d}$-local system, or étale (c, d)-$\mathbb{Q}_p$-local system on $\mathrm{Spec}(A_{\psi,\infty})$, equipped with an action of an open subgroup H of Γ_N. We say that this action is *effective* if for each $s \in M_\sigma$, the subgroup of $N_p \cap H$ fixing s also acts trivially on the pullback of E to $\mathrm{Spec}(A_{\psi,\infty}/(s, s^{1/p}, \ldots))$. We make similar definitions over $\mathcal{M}(A_{\psi,\infty})$. This condition is automatic when ψ is boundary-free.

Theorem 3.3.6. *For any open subgroup H of Γ_N, for any $c, d, m \in \mathbb{Z}$ with $d, m > 0$, the categories of locally constant étale $\mathbb{Z}/p^m\mathbb{Z}$-sheaves, étale $\mathbb{Z}_{p^d}$-local systems, étale $\mathbb{Q}_{p^d}$-local* *systems, and étale (c,d)-$\mathbb{Q}_p$-local systems over* $\operatorname{Spec}(A^H_{\psi,\infty})$ *are respectively equivalent to the categories of locally constant étale* $\mathbb{Z}/p^m\mathbb{Z}$*-sheaves, étale* $\mathbb{Z}_{p^d}$*-local systems, étale* $\mathbb{Q}_{p^d}$*-local systems, and étale* (c,d)*-*$\mathbb{Q}_p$*-local systems over* $\operatorname{Spec}(A_{\psi,\infty})$ *carrying continuous effective actions of* H*. The same is also true with* $\operatorname{Spec}(A^H_{\psi,\infty}), \operatorname{Spec}(A_{\psi,\infty})$ *replaced by* $\mathcal{M}(A^H_{\psi,\infty}), \mathcal{M}(A_{\psi,\infty})$*, respectively.*

Proof. It is enough to check the claim for locally constant étale $\mathbb{Z}/p^m\mathbb{Z}$-sheaves (noting that the reduction to this case requires arbitrary H even if we start with $H = \Gamma_N$). The functor from objects over $\operatorname{Spec}(A^H_{\psi,\infty})$ to objects over $\operatorname{Spec}(A_{\psi,\infty})$ equipped with continuous effective H-actions is just base extension, which we need to check is essentially surjective. We do this by induction primarily on $\operatorname{rank}(N)$ and secondarily on the dimension of the $\mathbb{R}$-span of σ. Note that by Remark 3.3.4, continuity of the action gives a descent to a sheaf over $\operatorname{Spec}(A^{H'}_{\psi,\infty})$ for some open subgroup H' of H.

If $\sigma = \{0\}$, then the effectivity condition is empty. The residual action of H/H' defines a descent datum; since $A^{H'}_{\psi,\infty}$ is faithfully finite étale over $A^H_{\psi,\infty}$, we may conclude by Theorem 3.3.1.

If $\sigma \neq \{0\}$, we can find some $s \in M_\sigma$ which is not invertible. Let I be the ideal of $A^{H'}_{\psi,\infty}$ generated by s^{1/p^n} for all nonnegative integers n for which $s^{1/p^n} \in A^{H'}_{\psi,\infty}$ (there are only finitely many such n). Put $B_1 = A^{H'}_{\psi,\infty}[s^{-1}]$ and let B_2 be the I-adic completion of $A^{H'}_{\psi,\infty}$. Then $\operatorname{Spec}(B_1 \oplus B_2) \to \operatorname{Spec}(A^{H'}_{\psi,\infty})$ is a covering for the fpqc topology, so by Theorem 3.3.1 again, it is enough to exhibit a descent datum for this covering. Note that we may replace B_2 by B_2/I since the base extension $\mathbf{FÉt}(B_2) \to \mathbf{FÉt}(B_2/I)$ is an equivalence [45, Theorem 1.2.8]. To conclude, we note that the induction hypothesis applies: B_1 corresponds to one case with the same N but smaller σ, while B_2/I corresponds to finitely many cases with smaller N plus some glueing on overlaps. □

3.4 Perfect period rings and (φ, Γ_N)-modules

In preparation for a description of étale local systems, we introduce a suite of period rings associated to the frame ψ and a notion of relative (φ, Γ)-modules. We characterize these period rings as *perfect* because the Frobenius map φ acts on them via isomorphisms; we will introduce a contrasting suite of *imperfect period rings* somewhat later.

Definition 3.4.1. For $0 < s \leq r$, put

$$\begin{gathered}
\tilde{\mathbf{A}}_\psi = W(\overline{A}_\psi), \qquad \tilde{\mathbf{A}}^{\dagger,r}_\psi = \tilde{\mathcal{R}}^{\mathrm{int},r}_{\overline{A}_\psi}, \qquad \tilde{\mathbf{A}}^{\dagger}_\psi = \tilde{\mathcal{R}}^{\mathrm{int}}_{\overline{A}_\psi}, \\
\tilde{\mathbf{B}}_\psi = \tilde{\mathbf{A}}_\psi[p^{-1}], \qquad \tilde{\mathbf{B}}^{\dagger,r}_\psi = \tilde{\mathbf{A}}^{\dagger,r}_\psi[p^{-1}], \qquad \tilde{\mathbf{B}}^{\dagger}_\psi = \tilde{\mathbf{A}}^{\dagger}_\psi[p^{-1}], \\
\tilde{\mathbf{C}}^{[s,r]}_\psi = \tilde{\mathcal{R}}^{[s,r]}_{\overline{A}_\psi}, \qquad \tilde{\mathbf{C}}^{r}_\psi = \tilde{\mathcal{R}}^{r}_{\overline{A}_\psi}, \qquad \tilde{\mathbf{C}}_\psi = \tilde{\mathcal{R}}_{\overline{A}_\psi}
\end{gathered}$$

in the sense of [45, Definition 5.1.1]. We will refer collectively to these rings (including $\overline{A}_\psi$) as the *perfect period rings* associated to ψ. All of these rings inherit bijective actions of φ and Γ_N, except that φ maps $\tilde{\mathbf{A}}^{\dagger,r}_\psi$ to $\tilde{\mathbf{A}}^{\dagger,r/p}_\psi$ and so forth.

Remark 3.4.2. Recall that the rings introduced in Definition 3.4.1 carry certain topologies described in [45, Definition 5.1.3]: the rings contained in $\tilde{\mathbf{B}}_\psi$ carry a p-adic topology and a weak topology, the rings contained in $\tilde{\mathbf{C}}_\psi^{[s,r]}$ or $\tilde{\mathbf{C}}_\psi^r$ carry a Fréchet topology, and the rings contained in $\tilde{\mathbf{C}}_\psi$ carry an LF (limit-of-Fréchet) topology. The actions of φ and Γ_N are continuous for all of these topologies *except* that Γ_N does not act continuously for the p-adic topology. (This would require the action of Γ_N on $\overline{A}_\psi$ to be trivial on some open subgroup, which it isn't.)

Lemma 3.4.3. *If A is connected, then $(\tilde{\mathbf{B}}_\psi)^{\varphi,\Gamma_N} = (\tilde{\mathbf{C}}_\psi)^{\varphi,\Gamma_N} = \mathbb{Q}_p$.*

Proof. By [45, Corollary 5.2.4],

$$(\tilde{\mathbf{B}}_\psi)^{\varphi,\Gamma_N} = (\tilde{\mathbf{C}}_\psi)^{\varphi,\Gamma_N} = W(\overline{A}_\psi^{\overline{\varphi},\Gamma_N})[p^{-1}].$$

Given $e \in \overline{A}_\psi^{\overline{\varphi},\Gamma_N}$, decompose e as $\sum_{i\in\mathbb{F}_p} ie_i$ as in [45, Lemma 3.1.2]; then each e_i is a Γ_N-invariant idempotent in $\overline{A}_\psi$. However, since A is connected, so is $\mathcal{M}(A)$ by Kiehl's theorem [45, Theorem 2.5.13(b)]. Hence $\mathcal{M}(\overline{A}_\psi) \cong \mathcal{M}(A_{\psi,\infty})$ admits no proper nonempty Γ_N-invariant closed and open subsets, so $\overline{A}_\psi$ has no Γ_N-invariant idempotents other than 0 or 1. We conclude that $e \in \mathbb{F}_p$, proving the claim. □

Definition 3.4.4. For d a positive integer, let M be a φ^d-module or local φ^d-module over a perfect period ring R equipped with a semilinear action of Γ_N on M commuting with φ^d. The Γ_N-action is *continuous* if the action map $\Gamma_N \times M \to M$ is continuous for all of the available topologies on R. (We omit the p-adic topology because the action is not even continuous on the ring R.) The action is *effective* if for each $s \in M_\sigma$, the subgroup of $N_p \cap H$ fixing s also acts trivially on $M/(s, s^{1/p}, \dots)M$. If both conditions are satisfied, we describe M as a *(φ^d, Γ_N)-module* or *local (φ^d, Γ_N)-module*; we say M is *pure*, *étale*, *globally pure*, or *globally étale* as a (local) (φ^d, Γ_N)-module if it is pure, étale, globally pure, or globally étale as a (local) φ^d-module (in the sense of [45, §7.3]). Note that we do not insist on any compatibility between Γ_N and pure models.

Example 3.4.5. The element

$$t = \log(1+\pi) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\pi^i}{i} \in \tilde{\mathbf{C}}_\psi$$

satisfies $\varphi(t) = pt$ and $\gamma(t) = \gamma \cdot t$ for $\gamma \in \mathbb{Z}_p^\times$ and is fixed by N_p. We may thus view $t^{-1}\tilde{\mathbf{C}}_\psi$ as a (φ, Γ_N)-module over $\tilde{\mathbf{C}}_\psi$ which is pure of slope 1. More generally, for M a (φ, Γ_N)-module over $\tilde{\mathbf{C}}_\psi$ and $m \in \mathbb{Z}$, we may view $t^{-m}M$ as a (φ, Γ_N)-module over $\tilde{\mathbf{C}}_\psi$; we also denote this (φ, Γ_N)-module by $M(m)$ and describe it as a *twist* of M.

We may formally factor

$$t = u\pi \prod_{i=1}^{\infty} \frac{(1+\pi)^{p^i} - 1}{(1+\pi)^{p^{i-1}} - 1}$$

for some unit $u \in \tilde{\mathbf{C}}_\psi$. For each positive integer d, put

$$t_d = \prod_{i=1}^{\infty} \frac{(1+\pi)^{p^{di}} - 1}{(1+\pi)^{p^{di-1}} - 1};$$

then $t_d^{-1}\tilde{\mathbf{C}}_\psi$ is a (φ^d, Γ_N)-module over $\tilde{\mathbf{C}}_\psi$ which is pure of slope $1/d$.

Remark 3.4.6. Let M be a (φ, Γ_N)-module over $\tilde{\mathbf{C}}_\psi$. As a function on $\mathcal{M}(\overline{A}_\psi) \cong \mathcal{M}(A_{\psi,\infty})$, the slope polygon of M is Γ_N-invariant; it thus descends to a map on $\mathcal{M}(A)$. Since the map $\mathcal{M}(A_{\psi,\infty}) \to \mathcal{M}(A)$ is a quotient map (see Definition 3.2.1), we may descend various assertions about the slope polygon from $\mathcal{M}(A_{\psi,\infty})$ to $\mathcal{M}(A)$, such as the following.

- The slope polygon function is bounded and lower semicontinuous on $\mathcal{M}(A)$ [45, Theorem 7.4.5, Proposition 7.4.6]. Consequently, the pure locus and the étale locus are open.
- There is an open dense subset of $\mathcal{M}(A)$ on which the slope polygon is locally constant [45, Corollary 7.4.7].
- If the slope polygon function is locally constant, then M admits a global slope filtration [45, Theorem 7.4.8].

Definition 3.4.7. Let d be a positive integer. For M a (φ^d, Γ_N)-module over any perfect period ring, start with the complex

$$0 \to M \overset{\varphi^d - 1}{\to} M \to 0, \tag{3.4.7.1}$$

then replace M in each position by the complex of continuous effective Γ_N-cochains with values in M. (A cochain, viewed as a function on Γ_N^i for some nonnegative integer i, is *effective* if for each $s \in M_\sigma$, for H the subgroup of N_p fixing s, elements of Γ_N^i which are termwise congruent modulo H map to elements of M which are congruent modulo $(s, s^{1/p}, \dots)$.) Let $H^i_{\varphi^d,\Gamma}(M)$ denote the total cohomology of the resulting double complex.

Remark 3.4.8. If ψ is a boundary-free frame, then $H^i_{\varphi^d,\Gamma}(M)$ is the i-th hypercohomology of the complex (3.4.7.1) induced by continuous Γ_N-cohomology (i.e., the functor of invariants on the category of topological abelian groups equipped with continuous actions of Γ_N).

Remark 3.4.9. Let $\psi' : \mathcal{M}(A') \to K(\sigma')$ be a second toric frame, and specify a morphism of toric frames as in Definition 2.3.1. We then obtain functoriality maps between each period ring $*_\psi$ and the corresponding period ring $*_{\psi'}$; these maps are continuous for all available topologies and equivariant with respect to φ. They are also equivariant for the Γ-actions in the following sense. The given toric morphism induces a $\mathbb{Z}_p$-linear homomorphism $N'_p \to N_p$ and hence a group homomorphism $\Gamma_{N'} \to \Gamma_N$. The functoriality maps are then equivariant for the action of $\Gamma_{N'}$ on $*_\psi$ via $\Gamma_{N'} \to \Gamma_N$ and the usual action of $\Gamma_{N'}$ on $*_{\psi'}$. We thus obtain base extension functors from (local) (φ, Γ_N)-modules on $*_\psi$ to (local) $(\varphi, \Gamma_{N'})$-modules on $*_{\psi'}$; these preserve the globally pure/étale and pure/étale conditions.

3.5 Boundedness and continuity of Γ-actions

In case ψ is an étale morphism, the continuity of the action of Γ_N in the definition of a (φ, Γ_N)-module can be replaced by a formally weaker condition, as follows.

Lemma 3.5.1. *Let G be a profinite group each of whose subgroups of finite index is open, and which admits a continuous isometric action on an analytic field F. Let E be a finite extension of F to which the action of G extends. Then the extended action is also continuous.*

Proof. Since the claim is evident for E contained in the direct perfection of F, we may reduce to the case where E/F is Galois. Given $x \in E$, let $P(T) = \prod_{z \in S}(T - z) \in F[T]$ be the minimal polynomial of x over F and put $n = \deg(P)$. For $\gamma \in G$, we then have

$$0 = \gamma(P(x)) = \gamma(P)(\gamma(x)) = (\gamma(P) - P)(\gamma(x)) + \prod_{z \in S}(\gamma(x) - z).$$

Choose $\epsilon > 0$ so that $|y - z| > \epsilon$ whenever y and z are distinct elements of S. We can then find an open subgroup G_1 of G such that for all $\gamma \in G_1$ and all $y \in S$, $|(\gamma(P) - P)(\gamma(y))| < \epsilon^n$. This means that $\prod_{z \in S} |\gamma(y) - z| < \epsilon^n$, so there exists $z = z(\gamma, y) \in S$ such that $|\gamma(y) - z| < \epsilon$; moreover, z is uniquely determined by γ and y. For $\gamma' \in G_1$, we have both $|\gamma'\gamma(y) - z(\gamma'\gamma, y)| < \epsilon$ and $|\gamma'(\gamma(y)) - \gamma'(z(\gamma, y))| < \epsilon$, forcing $z(\gamma'\gamma, y) = z(\gamma', z(\gamma, y))$. That is, the map z defines a group action of G_1 on S. The stabilizer of x under this group action is a subgroup G_2 of G_1 of finite index.

Since G_2 has finite index in G, it is open by hypothesis. For $\gamma \in G_2$, we have $|\gamma(x) - x| < \epsilon$ and so $|x - \gamma(w)| < \epsilon$ whenever $|x - w| < \epsilon$. This proves that the action on E is continuous. □

Remark 3.5.2. We will only apply Lemma 3.5.1 in cases where it is obvious that the subgroups of the profinite group G of finite index are all open. Nonetheless, it is worth pointing out that this is true whenever G is topologically finitely generated, by a theorem of Nikolov and Segal [54, 55].

Lemma 3.5.3. *Let G be a profinite group each of whose subgroups of finite index is open. Then for any positive integers n and d, any homomorphism $\tau : G \to \mathrm{GL}_n(\mathbb{Q}_{p^d})$ with bounded image is continuous.*

Proof. The closure H of the image of τ is a subgroup of the locally compact group $\mathrm{GL}_n(\mathbb{Q}_{p^d})$ which is closed and bounded, and hence compact. For each open subgroup J of H, $\tau^{-1}(J)$ is a subgroup of G of finite index, which is thus open. Hence τ is continuous. □

Remark 3.5.4. Note that the boundedness condition in Lemma 3.5.3 is needed to avoid pathologies such as the following. Choose a basis of $\mathbb{Q}_p$ as a $\mathbb{Q}$-vector space, then use this basis to define a $\mathbb{Q}$-linear projection $\tau : \mathbb{Q}_p \to \mathbb{Q}$. We then obtain a homomorphism $\mathbb{Z}_p \to \mathrm{GL}_2(\mathbb{Q}_p)$ which is not continuous by taking

$$t \mapsto \begin{pmatrix} 1 & \tau(t) \\ 0 & 1 \end{pmatrix}.$$

One obtains similar pathologies if one omits the boundedness condition in Proposition 3.5.7.

Definition 3.5.5. For M an abelian group equipped with a norm $|\cdot|$, an action of a group G on M is *bounded* if there exists $c > 0$ such that $|g(x)| \leq c|x|$ for all $g \in G, x \in M$. This condition evidently depends only on the equivalence class of the norm on M.

For R a perfect uniform Banach $\mathbb{F}_p$-algebra with norm α and M a φ^d-module over $\tilde{\mathcal{R}}_R$, we say that the action of G on M is *bounded* if for each r, s with $0 < s \leq r/p^d$, for some (and hence any) norm on the model $M_{[s,r]}$ of M over $\tilde{\mathcal{R}}_R^{[s,r]}$ corresponding to the norm $\max\{\lambda(\alpha^s), \lambda(\alpha^r)\}$ on $\tilde{\mathcal{R}}_R^{[s,r]}$ as in [45, Lemma 2.2.12], the action of G on $M_{[s,r]}$ is bounded as in the previous paragraph.

Lemma 3.5.6. *Let L be a perfect nontrivially normed analytic field of characteristic p. Let G be a profinite group each of whose subgroups of finite index is open, and suppose G acts continuously on L. For d a positive integer, let M be a φ^d-module over $\tilde{\mathcal{R}}_L$ admitting a semilinear action of G commuting with φ. Then the action of G on M is continuous if and only if it is bounded.*

Proof. Since G is compact, continuity evidently implies boundedness, so we need only check the converse. Suppose first that M is trivial. In this case, on any basis fixed by φ^d, G acts via matrices over $\tilde{\mathcal{R}}_L^{\varphi^d}$, which equals $\mathbb{Q}_{p^d}$ by [45, Corollary 5.2.4]. The resulting homomorphism $G \to \mathrm{GL}_n(\mathbb{Q}_{p^d})$ for $n = \mathrm{rank}(M)$ is continuous by Lemma 3.5.3, so G acts continuously on M as desired.

Suppose next that M is pure. There is no harm in enlarging d so that $d\mu(M) \in \mathbb{Z}$; then M corresponds via [45, Theorem 8.1.6] to an étale $\mathbb{Q}_{p^d}$-local system over $\mathrm{Spec}(L)$. It follows that we can exhibit a perfect analytic field L' which is the completion of a possibly infinite Galois extension of L, such that G extends to L' (necessarily continuously by Lemma 3.5.1) and $M \otimes_{\tilde{\mathcal{R}}_L} \tilde{\mathcal{R}}_{L'}$ admits a basis over $\tilde{\mathcal{R}}_{L'}$ on which φ^d acts by multiplication by p^{c_i} for some $c_i \in \mathbb{Z}$. By passing from L to L', we may thus reduce to the previous case.

To handle the general case, let $0 = M_0 \subset \cdots \subset M_l = M$ be the slope filtration of M provided by [45, Theorem 4.2.12]. Since this filtration is unique, it admits an action of G. By the previous paragraph, the induced action of G on each M_i/M_{i-1} is continuous. It follows easily that the action on M is also continuous. □

Proposition 3.5.7. *Suppose that ψ is étale. For d a positive integer, let M be a φ^d-module over $\tilde{\mathbf{C}}_\psi$ equipped with a semilinear action of Γ_N commuting with φ^d. Then this action is continuous if and only if it is bounded. (In particular, if the action is bounded and effective, then M is a (φ^d, Γ_N)-module.)*

Proof. The claim is local on $\mathcal{M}(A)$, so we may assume ψ is of rational type. In this case, by Theorem 3.1.7 and Lemma 3.1.10, $\overline{A}_\psi$ is the completed perfection of an affinoid algebra over $k((\overline{\pi}))$. Let Δ be the Shilov boundary of $\overline{A}_\psi$ (see Definition 1.1.2). Since Δ is finite, its elements are all fixed by some subgroup G of Γ_N of finite index. By making G a bit smaller, we can ensure that the points of $\mathcal{M}(R)$ above points of Δ are all fixed by G also. Since every subgroup of Γ_N of finite index is open, by Lemma 3.5.6, the action of G on $M \otimes_{\tilde{\mathbf{C}}_\psi} \tilde{\mathcal{R}}_{\mathcal{H}(\delta)}$ is continuous for each $\delta \in \Delta$. From the definition of the Shilov boundary, it follows that G acts continuously on M, as then does Γ_N. □

3.6 (φ, Γ_N)-modules and local systems

Using Theorem 3.3.6 (to pass to a deeply ramified cover) and the results of [45] (to relate characteristic 0 to characteristic p), we can now use perfect period rings to describe étale local systems and their étale cohomology.

Theorem 3.6.1. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, the following tensor categories are equivalent.*

(a) The category of étale $\mathbb{Z}_{p^d}$-local systems over $\operatorname{Spec}(A)$.

(b) The category of (φ^d, Γ_N)-modules over $\tilde{\mathbf{A}}_\psi$.

(c) The category of (φ^d, Γ_N)-modules over $\tilde{\mathbf{A}}^\dagger_\psi$.

More precisely, the functor from (c) to (b) is base extension.

Proof. The equivalence of (a) and (b) follows from Theorem 3.3.6 and [45, Theorem 8.1.2], as does full faithfulness of base extension from (c) to (b). To check essential surjectivity of this base extension, it suffices to check that for any φ^d-module $M^\dagger$ over $\tilde{\mathbf{A}}^\dagger_\psi$, if $M = M^\dagger \otimes_{\tilde{\mathbf{A}}^\dagger_\psi} \tilde{\mathbf{A}}_\psi$ admits a continuous effective Γ_N-action, then this action induces a continuous effective action on $M^\dagger$. We obtain an effective action on $M^\dagger$ by [45, Theorem 8.1.2], but it remains to check continuity since there is no LF topology on $\tilde{\mathbf{A}}_\psi$.

By [45, Theorem 8.1.2] again, the underlying φ^d-module of $M^\dagger$ corresponds to a $\mathbb{Z}_{p^d}$-local system $T = \{\cdots \to T_2 \to T_1\}$ on $\operatorname{Spec}(\overline{A}_\psi)$, which we may safely assume is of constant rank $m > 0$. For each positive integer n, let T_n be represented by $\overline{U}_n \in \mathbf{F\acute{E}t}(\overline{A}_\psi)$. Let $\overline{U}$ be the completed direct limit of the $\overline{U}_n$, and put $M_{\overline{U}} = M \otimes_{\tilde{\mathbf{A}}_\psi} W(\overline{U})$. We then have

$$M^\dagger = (M^\varphi_{\overline{U}} \otimes_{W(\overline{U})^\varphi} \tilde{\mathcal{R}}^{\mathrm{int}}_{\overline{U}})^{\mathrm{GL}_m(\mathbb{Z}_{p^d})}, \qquad M = (M^\varphi_{\overline{U}} \otimes_{W(\overline{U})^\varphi} W(\overline{U}))^{\mathrm{GL}_m(\mathbb{Z}_{p^d})}.$$

The action of Γ_N on $M^\varphi_{\overline{U}}$ is continuous for the weak topology, but on this set the p-adic, weak, and LF topologies coincide [45, Remark 5.1.5]. We thus deduce that the action of Γ_N on $M^\dagger$ is continuous for all topologies, as desired. □

Theorem 3.6.2. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, the following tensor categories are equivalent.*

(a) The category of étale (c,d)-$\mathbb{Q}_{p^d}$-local systems over $\operatorname{Spec}(A)$.

(b) The category of globally (c,d)-pure (φ^d, Γ_N)-modules over $\tilde{\mathbf{B}}_\psi$.

(c) The category of globally (c,d)-pure (φ^d, Γ_N)-modules over $\tilde{\mathbf{B}}^\dagger_\psi$.

(d) The category of globally (c,d)-pure (φ^d, Γ_N)-modules over $\tilde{\mathbf{C}}_\psi$.

More precisely, the functors from (c) to (b) and (d) are base extensions.

Proof. This follows from Theorem 3.3.6 and [45, Theorem 8.1.4] by arguing as in Theorem 3.6.1 to match up topologies. ☐

Theorem 3.6.3. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, the following tensor categories are equivalent.*

(a) The category of étale (c, d)-$\mathbb{Q}_{p^d}$-local systems over $\mathcal{M}(A)$.

(b) The category of (c, d)-pure local (φ^d, Γ_N)-modules over $\tilde{\mathbf{B}}_\psi$.

(c) The category of (c, d)-pure local (φ^d, Γ_N)-modules over $\tilde{\mathbf{B}}^\dagger_\psi$.

(d) The category of (c, d)-pure (φ^d, Γ_N)-modules over $\tilde{\mathbf{C}}_\psi$.

More precisely, the functors from (c) to (b) and (d) are base extensions.

Proof. This follows from Theorem 3.6.2 and the fact that φ^d-modules over $\tilde{\mathbf{C}}_\psi$ glue over covering families of rational localizations [45, Corollary 6.4.4]. ☐

Theorem 3.6.4. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, let T be an étale $\mathbb{Z}_{p^d}$-local system on* Spec(A). *Let M be the (φ^d, Γ_N)-module over one of $\tilde{\mathbf{A}}_\psi$ or $\tilde{\mathbf{A}}^\dagger_\psi$ corresponding to T via Theorem 3.6.1. Then for $i \geq 0$, there is a natural (in T and A) bijection $H^i_{\text{ét}}(\operatorname{Spec}(A), T) \cong H^i_{\varphi^d,\Gamma}(M)$.*

Proof. This follows immediately from [45, Theorem 8.2.1]. ☐

Theorem 3.6.5. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, let E be an étale $\mathbb{Q}_{p^d}$-local system on* Spec(A). *Let M be the globally étale (φ^d, Γ_N)-module over one of $\tilde{\mathbf{B}}_\psi, \tilde{\mathbf{B}}^\dagger_\psi, \tilde{\mathbf{C}}_\psi$ corresponding to E via Theorem 3.6.2. Then for $i \geq 0$, there is a natural (in E and A) bijection $H^i_{\text{ét}}(\operatorname{Spec}(A), E) \cong H^i_{\varphi^d,\Gamma}(M)$.*

Proof. This follows from [45, Theorem 8.2.3]. ☐

Theorem 3.6.6. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, let E be an étale $\mathbb{Q}_{p^d}$-local system on $\mathcal{M}(A)$. Let M be the étale (φ^d, Γ_N)-module over $\tilde{\mathbf{C}}_\psi$ corresponding to E via Theorem 3.6.3. Then for $i \geq 0$, there is a natural (in E and A) bijection $H^i_{\text{ét}}(\mathcal{M}(A), E) \cong H^i_{\varphi^d,\Gamma}(M)$.*

Proof. This follows from [45, Theorem 8.3.6]. ☐

3.7 (φ, Γ)-modules, vector bundles, and B-pairs

We can interpret (φ, Γ_N)-modules over $\tilde{\mathbf{C}}_\psi$ in the language of vector bundles on certain schemes; this generalizes a result of Fargues-Fontaine for $A = K$ [26]. We also obtain a similar description in the language of B-pairs as introduced by Berger [10].

Definition 3.7.1. Write P_ψ for the graded ring $P_{\overline{A}_\psi}$ of [45, Definition 6.3.1]. By a Γ_N-*vector bundle* on $\operatorname{Proj}(P_\psi)$, we mean a quasicoherent finite locally free sheaf on $\operatorname{Proj}(P_\psi)$ equipped with a continuous (as in [45, Definition 6.3.17]) effective action on Γ_N.

Definition 3.7.2. Put $\mathbf{B}_{\mathrm{e},\psi} = P_\psi[t^{-1}]_0$. Let $\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi}$ be the π-adic completion of $\tilde{\mathbf{B}}^{\dagger,1}_\psi$; note that $\operatorname{Spec}(\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi})$ may be naturally identified with the t-adic completion of $\operatorname{Proj}(P_\psi)$. Put $\mathbf{B}^{\nabla}_{\mathrm{dR},\psi} = \mathbf{B}^{\nabla+}_{\mathrm{dR},\psi}[t^{-1}]$; this ring receives natural maps from both $\mathbf{B}_{\mathrm{e},\psi}$ and $\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi}$. All of these rings inherit actions of φ and Γ_N.

By a *B-pair* over ψ, we will mean a triple $M = (M_\mathrm{e}, M^{\nabla+}_{\mathrm{dR}}, i)$ in which M_e is a finite projective module over $\mathbf{B}_{\mathrm{e},\psi}$ equipped with a continuous effective action of Γ_N, $M^{\nabla+}_{\mathrm{dR}}$ is a finite projective module over $\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi}$ equipped with a continuous effective action of Γ_N, and

$$i : M_\mathrm{e} \otimes_{\mathbf{B}_{\mathrm{e},\psi}} \mathbf{B}^{\nabla}_{\mathrm{dR},\psi} \cong M^{\nabla+}_{\mathrm{dR}} \otimes_{\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi}} \mathbf{B}^{\nabla}_{\mathrm{dR},\psi}$$

is a Γ_N-equivariant isomorphism. We denote the object defined by this isomorphism by M^{∇}_{dR}.

Theorem 3.7.3. *The following categories are naturally (in ψ) equivalent.*

(a) The category of (φ, Γ_N)-modules over $\tilde{\mathbf{C}}_\psi$.

(b) The category of Γ_N-vector bundles on $\operatorname{Proj}(P_\psi)$*.*

(c) The category of B-pairs over ψ.

Proof. The equivalence between (a) and (b) follows from [45, Theorem 6.3.12]. The equivalence between (b) and (c) follows from [45, Proposition 8.3.11]. □

Remark 3.7.4. The definition of B-pairs amounts to using a covering of $\operatorname{Proj}(P_\psi)$ in the flat topology rather than the Zariski topology (modulo the fact that the map $\operatorname{Spec}(\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi}) \to \operatorname{Proj}(P_\psi)$ is not in general known to be flat). This covering has the convenient feature of being Γ_N-stable; we use this feature next to interpret cohomology of (φ, Γ)-modules in terms of vector bundles. This is necessary to work around the fact that it is tricky to make sense of equivariant coherent cohomology of sheaves on $\operatorname{Proj}(P_\psi)$ with continuous action of Γ_N, because Γ_N does not act continuously on $\operatorname{Proj}(P_\psi)$.

Definition 3.7.5. Let V be a Γ_N-vector bundle on $\operatorname{Proj}(P_\psi)$. For $j \geq 0$, let $C^{0,j}$ (resp. $C^{1,j}$) be the set of continuous effective j-cochains for Γ_N with values in $\Gamma(\operatorname{Spec}(\mathbf{B}_{\mathrm{e},\psi}) \cup \operatorname{Spec}(\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi}), V)$ (resp. $\Gamma(\operatorname{Spec}(\mathbf{B}^{\nabla}_{\mathrm{dR},\psi}), V)$). The $C^{i,j}$ form a double complex (using Čech differentials along i and cochain differentials along j), whose total cohomology we call $H^i(\operatorname{Proj}(P_\psi), \Gamma_N; V)$.

Theorem 3.7.6. *Let V be the Γ_N-vector bundle on* $\operatorname{Proj}(P_\psi)$ *corresponding to the (φ, Γ_N)-module M over $\tilde{\mathbf{C}}_\psi$ via Theorem 3.7.3. Then for $i \geq 0$, there is a natural (in V and ψ) bijection $H^i(\operatorname{Proj}(P_\psi), \Gamma_N; V) \cong H^i_{\varphi,\Gamma}(M)$.*

Proof. This follows at once from [45, Theorem 8.3.3] and the fact that one may compute cohomology of quasicoherent sheaves on $\mathrm{Proj}(P_\psi)$ using the Čech complex for the covering by $\mathrm{Spec}(\mathbf{B}_{\mathrm{e},\psi})$ and $\mathrm{Spec}(\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi})$ [45, Proposition 8.3.11]. $\square$

Remark 3.7.7. As in [45, Remark 8.3.8], we may reinterpret Theorem 3.6.6 as follows. Let E be an étale $\mathbb{Q}_p$-local system on $\mathcal{M}(A)$. Associate to E an étale (φ, Γ_N)-module M over $\tilde{\mathbf{C}}_\psi$ via Theorem 4.8.4, then associate to M a Γ_N-vector bundle V on $\mathrm{Proj}(P_\psi)$ via Theorem 3.7.3. By Theorem 3.6.6 and Theorem 3.7.6, for $i \geq 0$, we have natural (in E and A) bijections $H^i_{\text{ét}}(\mathcal{M}(A), E) \cong H^i(\mathrm{Proj}(P_\psi), \Gamma_N; V)$.

Remark 3.7.8. The symbol ∇ in the notation for the rings $\mathbf{B}^{\nabla+}_{\mathrm{dR},\psi}, \mathbf{B}^{\nabla}_{\mathrm{dR},\psi}$ indicates that these should be considered as the kernels of integrable connections on certain larger rings. Such larger rings appear in the work of Scholze [63]; we will encounter them later in this series.

3.8 Compatibility with toric refinements

In preparation for globalization of the construction, we consider the interaction of relative (φ, Γ)-modules with toric refinements.

Hypothesis 3.8.1. Throughout §3.8, consider a toric refinement of toric frames with notation as in (2.3.1.1). Let H be the kernel of the map $\Gamma_{N'} \to \Gamma_N$ described in Remark 3.4.9. Let d be any positive integer.

Lemma 3.8.2. *For any $0 < s \leq r$, for*

$$* = \tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{A}}^{\dagger,r}, \tilde{\mathbf{B}}, \tilde{\mathbf{B}}^\dagger, \tilde{\mathbf{B}}^{\dagger,r}, \tilde{\mathbf{C}}, \tilde{\mathbf{C}}^+, \tilde{\mathbf{C}}^r, \tilde{\mathbf{C}}^{[s,r]},$$

the functoriality homomorphism induces an isomorphism $*_\psi \to *^H_{\psi'}$, and $*_{\psi'}$ is acyclic for continuous effective H-cochains.

Proof. For each positive integer n, let R_n be the subring of $\overline{A}_{\psi'}$ generated by $\overline{A}_\psi$ and s^{1/p^n} for $s \in T$. By [45, Theorem 3.6.12], R_n is the subring of $\overline{A}_{\psi'}$ fixed by the subgroup $p^n H$ of H, and becomes faithfully finite étale over $\overline{A}_\psi$ after inverting T. In particular, R_n is acyclic for effective $H/p^n H$-cochains by faithfully flat descent (Theorem 3.3.1). The claim then follows by writing $\overline{A}_{\psi'}$ as the completed direct limit of the R_n. $\square$

Definition 3.8.3. Define a *(φ^d, H)-module* over a perfect period ring $*_{\psi'}$ to be a φ^d-module equipped with a compatible continuous effective action of H. In particular, any $(\varphi^d, \Gamma_{N'})$-module may be viewed as a (φ^d, H)-module. By analogy with the cohomology of (φ^d, Γ_N)-modules (see Definition 3.4.7), we associate to a (φ^d, H)-module M the cohomology groups $H^i_{\varphi^d,H}(M)$ for $i \geq 0$; these vanish for $i > 1 + \mathrm{rank}(H)$.

Theorem 3.8.4. *For $* = \tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{B}}, \tilde{\mathbf{B}}^\dagger, \tilde{\mathbf{C}}$, the following results hold.*

(a) Base extension from φ^d-modules from $*_\psi$ to (φ^d, H)-modules over $*_{\psi'}$ is an equivalence of categories.

(b) *Base extension from* (φ^d, Γ_N)*-modules from* $*_\psi$ *to* $(\varphi^d, \Gamma_{N'})$*-modules over* $*_{\psi'}$ *is an equivalence of categories.*

Proof. Both parts follow from Lemma 3.8.2, with the quasi-inverse functor being the functor of H-invariants. $\square$

Theorem 3.8.5. *For* $* = \tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{B}}, \tilde{\mathbf{B}}^\dagger, \tilde{\mathbf{C}}$*, the following results hold.*

(a) *Let* M *be a* (φ^d, H)*-module over* $*_\psi$*. Then the natural maps* $H^i_{\varphi^d}(M) \to H^i_{\varphi^d,H}(M \otimes_{*_\psi} *_{\psi'})$ *are bijections for all* $i \geq 0$*.*

(b) *Let* M *be a* (φ^d, Γ_N)*-module over* $*_\psi$*. Then the natural maps* $H^i_{\varphi^d,\Gamma}(M) \to H^i_{\varphi^d,\Gamma}(M \otimes_{*_\psi} *_{\psi'})$ *are bijections for all* $i \geq 0$*.*

Proof. This again follows from Lemma 3.8.2. $\square$

3.9 Globalization

We conclude this discussion by describing relative (φ, Γ)-modules (of type $\tilde{\mathbf{C}}$) over an arbitrary K-analytic space. We globalize using the following construction.

Definition 3.9.1. Let X be a reduced K-analytic space. Define the *(boundary-free) framed affinoid site* of X to be the site consisting of the following data.

- The objects of the underlying category are pairs (j, ψ), where $j : \mathcal{M}(A) \to X$ is an affinoid subdomain and $\psi : \mathcal{M}(A) \to K(\sigma)$ is a (boundary-free) toric frame.
- Morphisms between two pairs $(j_1 : \mathcal{M}(A_1) \to X, \psi_1 : \mathcal{M}(A_1) \to K(\sigma_1))$ and $(j_2 : \mathcal{M}(A_2) \to X, \psi_2 : \mathcal{M}(A_2) \to K(\sigma_2))$ exist only when j_1 factors through j_2. In this case, any factorization corresponds to a morphism $\mathcal{M}(A_1) \to \mathcal{M}(A_2)$ of K-affinoid spaces, and the morphisms in the category consist of toric morphisms from ψ_1 to ψ_2 lifting such a factorization.
- A family of morphisms to a common target is a covering family if the underlying morphisms of affinoid spaces form a covering family of affinoid subdomains.

Remark 3.9.2. The sites constructed in Definition 3.9.1 are analogous to the G-topology introduced in Definition 1.2.7. One can make analogous definitions corresponding to the étale topology, but we will not need these here.

Definition 3.9.3. Let X be a K-analytic space. We define a *relative* (φ, Γ)*-module* over X to be an assignment to each pair (j, ψ) in the framed affinoid site of X a (φ, Γ_N)-module M_ψ over $\tilde{\mathbf{C}}_\psi$ and to each morphism $(j_1, \psi_1) \to (j_2, \psi_2)$ an isomorphism $M_{\psi_2} \otimes_{\tilde{\mathbf{C}}_{\psi_2}} \tilde{\mathbf{C}}_{\psi_1} \cong M_{\psi_1}$ of (φ, Γ_{N_1})-modules, satisfying the cocycle condition and the sheaf property for covering families of rational localizations. (In other words, we are specifying a *crystal* of (φ, Γ_N)-modules.) Note that it is equivalent to use the boundary-free framed affinoid site in this construction, by Theorem 3.8.4. For $f : Y \to X$ a morphism of K-analytic spaces, we obtain a pullback functor f^* from relative (φ, Γ)-modules over X to relative (φ, Γ)-modules over Y.

Theorem 3.9.4. *(a) For any frame $\psi : \mathcal{M}(A) \to K(\sigma)$, restriction to* (id, ψ) *defines an equivalence of categories between relative (φ, Γ)-modules over $\mathcal{M}(A)$ and (φ, Γ_N)-modules on $\tilde{\mathbf{C}}_\psi$.*

(b) Any admissible covering of a K-analytic space gives rise to an effective descent morphism for relative (φ, Γ)-modules.

Proof. Part (a) follows from Theorem 3.8.4. Given (a), (b) reduces (using Theorem 1.1.4) to the case of a covering of a single frame by rational subframes, for which we may apply [45, Corollary 6.4.4]. □

Definition 3.9.5. Let X be a K-analytic space. Let M be a relative (φ, Γ)-module on X. We define the cohomology $H^i_{\varphi,\Gamma}(M)$ of M as follows. Choose an admissible covering $\{(j_i, \psi_i)\}_{i \in I}$ of X in the framed affinoid site. For each finite fibre product of terms in the covering, form the complex of continuous effective Γ-cochains, then assemble these into a double complex using Čech differentials. The cohomology of the resulting site does not depend on the choice of the covering: this reduces to the case of a single frame being covered by finitely many rational subframes, in which case [45, Theorem 5.3.2, Proposition 6.4.5] implies the claim.

Remark 3.9.6. For X a K-analytic space, the existence of a resolution of singularities $f : Y \to X$ follows from Temkin's functorial desingularization of quasiexcellent $\mathbb{Q}$-schemes [67, 68]. We expect that the category of relative (φ, Γ)-modules over X is equivalent to the category of descent data in the category of relative (φ, Γ)-modules over Y (i.e., objects over Y equipped with isomorphisms of their two pullbacks to $Y \times_X Y$ satisfying the cocycle condition). If so, then one can use simplicial hypercoverings by smooth K-analytic spaces to study relative (φ, Γ)-modules over arbitrary K-analytic spaces.

Remark 3.9.7. An alternate approach to globalization can be obtained using Scholze's definition of the *pro-étale site* associated to an analytic space [63]. This approach has several technical advantages: it absorbs various continuity conditions in such a way that one can typically work with true sheaves rather than inverse systems, and it allows for the perfect period rings to be viewed as sheaves in a natural way. We will incorporate Scholze's viewpoint at a later stage in this series of papers.

4 Imperfect period rings

While the perfect period rings are sufficient for many purposes (e.g., the construction of universal local systems on Rapoport-Zink period domains), they suffer from some defects that cause problems for some applications; notably, one cannot differentiate the Γ_N-action on them (Remark 3.2.2). We now tackle the subtle task of defining a suite of *imperfect period rings*, on which φ does not act bijectively. We can only define these under some technical hypotheses; notably, we must assume A is a strictly K-affinoid algebra and that ψ is étale, not just unramified. As a result, imperfect period rings seem to be most relevant for studying (φ, Γ)-modules over smooth strictly K-analytic spaces. Moreover, we cannot discuss functoriality directly in terms of imperfect period rings, because the étaleness of ψ is not preserved by toric refinements. (Again, historical context has been reserved to §4.10.)

Hypothesis 4.0.1. For the remainder of the paper, assume unless otherwise specified that the frame ψ is of strictly rational type. In particular, A is a strictly K-affinoid algebra and ψ is an étale morphism.

Remark 4.0.2. The restriction in Hypothesis 4.0.1 that ψ be of rational type is needed to establish certain structural properties of imperfect period rings (§4.1). In case ψ is only étale, one can still get some information by working locally using Remark 1.2.11; see Theorem 4.8.4. It is somewhat more complicated to lift the restriction that A be strictly K-affinoid; see Remark 4.1.7.

4.1 Imperfect period rings and reality checks

Definition 4.1.1. We begin by constructing a p-adically saturated and complete subring $\mathbf{A}_\psi$ of $\tilde{\mathbf{A}}_\psi$ such that $\mathbf{A}_\psi/(p)$ is a strictly affinoid algebra over $k((\overline{\pi}))$ whose direct perfection is dense in $\overline{A}_\psi$. In case $A = K\{M_\sigma\}_\lambda$ for some $\lambda \in N_{\mathbb{Q}}$, we take $\mathbf{A}_\psi$ to be the weak completion of $K[\pi^{\pm 1}, M_\sigma]$. In case ψ is a strictly rational subdomain, we lift the description of $\overline{A}_\psi$ given by Lemma 3.1.10. In the general case, use the fact that once $\mathbf{A}_\psi$ is constructed for a given ψ, then $\mathbf{F\acute{E}t}(\mathbf{A}_\psi/(p)) \cong \mathbf{F\acute{E}t}(\overline{A}_\psi)$ by [45, Theorem 3.1.14].

Put $\mathbf{A}_\psi^{\dagger,r} = \mathbf{A}_\psi \cap \tilde{\mathbf{A}}_\psi^{\dagger,r}$ within $\tilde{\mathbf{A}}_\psi$, and put $\mathbf{A}_\psi^\dagger = \cup_{r>0} \mathbf{A}_\psi^{\dagger,r} = \mathbf{A}_\psi \cap \tilde{\mathbf{A}}_\psi^\dagger$. The pair $(\mathbf{A}_\psi, (p))$ is obviously henselian; one checks that $(\mathbf{A}_\psi^\dagger, (p))$ is also henselian by imitating the proof of [45, Proposition 5.5.3(a)].

Define $\mathbf{B}_\psi, \mathbf{B}_\psi^{\dagger,r}, \mathbf{B}_\psi^\dagger$ by inverting p in the corresponding rings of type $\mathbf{A}$. Note that $\mathbf{B}_\psi^{\dagger,r} = \mathbf{B}_\psi \cap \tilde{\mathbf{B}}_\psi^{\dagger,r}$ and $\mathbf{B}_\psi^\dagger = \mathbf{B}_\psi \cap \tilde{\mathbf{B}}_\psi^\dagger$ within $\tilde{\mathbf{B}}_\psi$.

Let $\mathbf{C}_\psi^{[s,r]}$ be the Fréchet completion of $\mathbf{B}_\psi^{\dagger,r}$ with respect to $\lambda(\overline{\alpha}_\psi^t)$ for $t \in [s,r]$; it embeds naturally into $\tilde{\mathbf{C}}_\psi^{[s,r]}$. Put $\mathbf{C}_\psi^r = \cap_{0<s\leq r} \mathbf{C}_\psi^{[s,r]}$ and $\mathbf{C}_\psi = \cup_{r>0} \mathbf{C}_\psi^r$.

We refer collectively to the rings

$$\mathbf{A}_\psi, \mathbf{A}_\psi/(p), \mathbf{A}_\psi^{\dagger,r}, \mathbf{A}_\psi^\dagger, \mathbf{B}_\psi, \mathbf{B}_\psi^{\dagger,r}, \mathbf{B}_\psi^\dagger, \mathbf{C}_\psi^{[s,r]}, \mathbf{C}_\psi^r, \mathbf{C}_\psi$$

as the *imperfect period rings* associated to ψ. (We need not include $\mathbf{A}_\psi^\dagger/(p)$ separately, because it coincides with $\mathbf{A}_\psi/(p)$.) All of these rings inherit actions of φ and Γ_N (with the same proviso about φ and r as in Definition 3.4.1).

Remark 4.1.2. Each imperfect ring embeds into its perfect counterpart, and thus inherits some topologies (see Remark 3.4.2). Since the actions of φ and Γ_N are continuous on the perfect period rings (excluding the action of Γ_N for the p-adic topology), the same is true of the imperfect period rings. In fact, somewhat more is true; see Proposition 4.3.7.

We need some *reality checks* in the style of [45, §5.2] but more in the spirit of [41, §2], where the condition of *having enough units* plays a key role.

Lemma 4.1.3. *There exists $r_0 > 0$ such that for $r \in (0, r_0]$, every $\overline{x} \in \mathbf{A}_\psi/(p)$ admits a lift $x \in \mathbf{A}_\psi^{\dagger,r}$ for which*

$$\lambda(\overline{\alpha}_\psi^r)(x - [\overline{x}]) \leq p^{-1/2}\lambda(\overline{\alpha}_\psi^r)(x).$$

In particular, $\lambda(\overline{\alpha}_\psi^r)(x) = \overline{\alpha}_\psi(\overline{x})^r$.

The key point here is that r_0 may be chosen uniformly in $\overline{x}$.

Proof. We first construct lifts with $\lambda(\overline{\alpha}_\psi^r)(x) \leq c\overline{\alpha}_\psi(\overline{x})^r$ for some $c > 0$ (independent of x). When $A = K\{M_\sigma\}_\lambda$ for some $\lambda \in N_{\mathbb{Q}}$, this can be trivially done with $c = 1$. In the general case, the homomorphism $k((\overline{\pi}))\{M_\sigma\}_\lambda \to \mathbf{A}_\psi/(p)$ is strictly affinoid, so there exists a strict surjection $k((\overline{\pi}))\{M_\sigma\}_\lambda\{T_1, \dots, T_m\} \to \mathbf{A}_\psi/(p)$ for some nonnegative integer m. We thus obtain the desired lifts by first lifting to $k((\overline{\pi}))\{M_\sigma\}_\lambda\{T_1, \dots, T_m\}$, then lifting each T_i to T_i (and lifting elements of $k((\overline{\pi}))\{M_\sigma\}_\lambda$ as before).

For some $r_1 > 0$, we can now construct lifts with $\lambda(\overline{\alpha}_\psi^r)(x) \leq c\overline{\alpha}_\psi(\overline{x})^r$ whenever $r \leq r_1$. For any sufficiently small $r_0 > 0$, we have $c^{r_0/r_1}p^{r_0/r_1 - 1} \leq p^{-1/2}$. Choose some such r_0; by [45, Lemma 5.2.1], for $r \in (0, r_0]$,

$$\begin{aligned}
\lambda(\overline{\alpha}_\psi^r)(x - [\overline{x}]) &\leq p^{r_0/r_1 - 1}\lambda(\overline{\alpha}_\psi^{r_1 r/r_0})(x - [\overline{x}])^{r_1/r_0} \\
&\leq p^{r_0/r_1 - 1}c^{r_1/r_0}\overline{\alpha}_\psi^r(\overline{x}) \\
&\leq p^{-1/2}\overline{\alpha}_\psi^r(\overline{x}).
\end{aligned}$$

This yields the claim. □

Corollary 4.1.4. *For r_0 as in Lemma 4.1.3, for $r \in (0, r_0]$, any $x \in \mathbf{B}_\psi^{\dagger,r}$ can be written as a $\lambda(\overline{\alpha}_\psi^r)$-convergent sum $\sum_{n=m}^\infty p^n y_n$ with $y_n \in \mathbf{A}_\psi^{\dagger,r_0}$ such that, if we write $\overline{y}_n$ for the image of y_n in $\mathbf{A}_\psi/(p)$, then $\lambda(\overline{\alpha}_\psi^{r_0})(y_n - [\overline{y}_n]) \leq p^{-1/2}\lambda(\overline{\alpha}_\psi^{r_0})(y_n)$ for all $n \geq 0$. Moreover, for any such representation, for all $s \in (0, r]$,*

$$\lambda(\overline{\alpha}_\psi^s)(x) = \max_n\{p^{-n}\overline{\alpha}_\psi^s(\overline{y}_n)\}. \tag{4.1.4.1}$$

Proof. We may assume $x \in \mathbf{A}_\psi^{\dagger,r}$. The existence of such a representation within $\mathbf{A}_\psi$ is immediate from Lemma 4.1.3. To prove convergence with respect to $\lambda(\overline{\alpha}_\psi^r)$, write $x = \sum_{n=0}^\infty p^n[\overline{x}_n]$ and note that $\lambda(\overline{\alpha}_\psi^r)(p^n y_n)$ is bounded above by the greater of $\lambda(\overline{\alpha}_\psi^r)(p^n[\overline{x}_n])$ and the maximum of $p^{-1/2}\lambda(\overline{\alpha}_\psi^r)(p^i y_i)$ over $i < n$. By an easy induction argument, it follows that there exists n such that

$$\lambda(\overline{\alpha}_\psi^r)\left(x - \sum_{i=0}^{n-1} p^i y_i\right) \leq p^{-1/2}\lambda(\overline{\alpha}_\psi^r)(x).$$

Applying this argument repeatedly yields the desired convergence, which in turn immediately implies (4.1.4.1). □

Lemma 4.1.5. *There exists $r_0 > 0$ such that for $0 < s \leq r \leq r_0$, $\mathbf{C}_\psi^{[s,r]}$ is a strictly affinoid algebra over K.*

Proof. Using the fact that $(\mathbf{A}_\psi^\dagger, (p))$ is a henselian pair, we may reduce to the case where ψ is a strictly rational subdomain. In this case, we take r_0 as in Lemma 4.1.3 and argue as follows. Set notation as in Lemma 3.1.10 with $p_1 = \cdots = p_m = 1$. Lift $\overline{f}_i, \overline{g} \in k[\overline{\pi}][M_\sigma]$ to $f_i, g \in \mathbf{A}_\psi^{\dagger,r_0}$ as per Lemma 4.1.3. Then for $0 < s \leq r \leq r_0$, $\mathbf{C}_\psi^{[s,r]}$ represents the strictly rational subspace

$$\{\beta \in K(\sigma) \times_K \mathcal{M}(K\{\pi\}) : \beta(\pi) \in [\omega^s, \omega^r], \beta(f_1) \leq \beta(g), \dots, \beta(f_m) \leq \beta(g)\}$$

of $K(\sigma) \times_K \mathcal{M}(K\{\pi\})$. □

Corollary 4.1.6. *For $r_0 > 0$ as in Lemma 4.1.5, for $0 < s \leq s' \leq r \leq r' \leq r_0$, inside $\mathbf{C}_\psi^{[s',r]}$ we have*

$$\mathbf{C}_\psi^{[s,r]} \cap \mathbf{C}_\psi^{[s',r']} = \mathbf{C}_\psi^{[s,r']}.$$

Proof. By Lemma 4.1.5, all of the rings in question are affinoid algebras over K. The claim then follows from Tate's theorem [45, Theorem 2.5.13(a)]. □

Remark 4.1.7. There is no analogue of Lemma 4.1.5 when ψ is rational but not strictly rational. For example, take ψ to be defined by the homomorphism $K\{T\} \to K\{T/q\}$ for some $q \in (0,1) \setminus p^{\mathbb{Q}}$. In this case,

$$\{\beta \in K(\sigma) \times_K \mathcal{M}(K\{\pi\}) : \beta(\pi) \in [\omega^s, \omega^r], \beta(T) \leq \beta(\pi)^{\log_\omega q}\}$$

is not a rational subspace of $K(\sigma) \times_K \mathcal{M}(K\{\pi\})$.

As a result, we are forced to assume that A is strictly K-affinoid in order to use the theorems of Tate and Kiehl. It may be possible to extend these to the cases that occur when A is not strictly K-affinoid, but we did not look into this. (The main difficulty is to extend Tate's theorem, as then the relevant cases of Kiehl's theorem would follow from [45, Proposition 2.7.5].)

Using the lifts from Lemma 4.1.3 in place of Teichmüller lifts, we obtain analogues of [45, Lemmas 5.2.5 and 5.2.7].

Lemma 4.1.8. *For r_0 as in Lemma 4.1.3, for $0 < s \leq r \leq r_0$, within $\mathbf{C}_\psi^{[s,s]}$ we have $\tilde{\mathbf{A}}_\psi^{\dagger,s} \cap \mathbf{C}_\psi^{[s,r]} = \mathbf{A}_\psi^{\dagger,r}$.*

Proof. Take x in the intersection, and write x as the limit in $\mathbf{C}_\psi^{[s,r]}$ of a sequence $x_0, x_1, \dots$ with $x_i \in \mathbf{B}_\psi^{\dagger,r}$. For each positive integer j, we can find $N_j > 0$ such that

$$\lambda(\overline{\alpha}_\psi^t)(x_i - x) \leq p^{-j} \qquad (i \geq N_j, t \in [s,r]).$$

Write $x_i = \sum_{l=m}^\infty p^l x_{il}$ as in Corollary 4.1.4. Put $y_i = \sum_{l=0}^\infty p^l x_{il} \in \mathbf{A}_\psi^{\dagger,r}$. For $i \geq N_j$, having $x \in \tilde{\mathbf{A}}_\psi^{\dagger,s}$ and $\lambda(\overline{\alpha}_\psi^s)(x_i - x) \leq p^{-j}$ implies that $\lambda(\overline{\alpha}_\psi^s)(p^l x_{il}) \leq p^{-j}$ for $l < 0$ by (4.1.4.1). That is,

$$\overline{\alpha}_\psi(\overline{x}_{il}) \leq p^{(l-j)/s} \qquad (i \geq N_j, l < 0).$$

Since $p^{-l} p^{(l-j)r/s} \leq p^{1+(1-j)r/s}$ for $l \leq -1$, we deduce that $\lambda(\overline{\alpha}_\psi^r)(x_i - y_i) \leq p^{1+(1-j)r/s}$ for $i \geq N_j$. Consequently, the sequence $y_0, y_1, \dots$ converges to x under $\lambda(\overline{\alpha}_\psi^r)$; it follows that $x \in \mathbf{A}_\psi^{\dagger,r}$ as desired. □

Lemma 4.1.9. *For r_0 as in Lemma 4.1.3, for $0 < s \leq r \leq r_0$, each $x \in \mathbf{C}_\psi^{[s,s]}$ can be decomposed as $y + z$ with $y \in \mathbf{A}_\psi^{\dagger,s}$, $z \in \mathbf{C}_\psi^{[s,r]}$, and*

$$\lambda(\overline{\alpha}_\psi^t)(z) \leq p^{1-t/s}\lambda(\overline{\alpha}_\psi^s)(x)^{t/s} \qquad (t \in [s,r]).$$

Proof. As in the proof of [45, Lemma 5.2.7], we may reduce to the case $x \in \mathbf{B}_\psi^{\dagger,s}$. Write $x = \sum_{n=m}^\infty p^n y_n$ as in Corollary 4.1.4, and put $y = \sum_{n=0}^\infty p^n y_n$ and $z = x - y$. For $n < 0$, we then have $\lambda(\overline{\alpha}_\psi^s)(p^n x_n) \leq \lambda(\overline{\alpha}_\psi^s)(x)$ and so

$$\begin{aligned}
\lambda(\overline{\alpha}_\psi^t)(p^n x_n) &= \lambda(\overline{\alpha}_\psi^t)(p^n[\overline{x}_n]) \\
&= p^{(-n)(1-t/s)}\lambda(\overline{\alpha}_\psi^s)(p^n x_n)^{t/s} \\
&\leq p^{1-t/s}\lambda(\overline{\alpha}_\psi^s)(x)^{t/s}.
\end{aligned}$$

This proves the claim. □

4.2 Period rings and étale covers

One of the key results from [45] making it possible to relate (φ, Γ_N)-modules over perfect period rings to étale local systems is the matching of certain finite étale algebras in characteristic 0 and characteristic p [45, Theorem 3.6.12, Corollary 5.5.6]. For imperfect period rings, we need similar results, but fortunately these are easily deduced from the perfect case.

Definition 4.2.1. Let $\widehat{\cup}\varphi^{-n}(\mathbf{A}_\psi)$ denote the p-adic completion of $\cup\varphi^{-n}(\mathbf{A}_\psi)$ within $\tilde{\mathbf{A}}_\psi$. For $r > 0$, let $\widehat{\cup}\varphi^{-n}(\mathbf{A}_\psi^{\dagger,p^{-n}r})$ denote the completion of $\cup\varphi^{-n}(\mathbf{A}_\psi^{\dagger,p^{-n}r})$ for the supremum of $\lambda(\overline{\alpha}_\psi^r)$ and the p-adic norm. Let $\widehat{\cup}\varphi^{-n}(\mathbf{A}_\psi^\dagger)$ denote the union of $\widehat{\cup}\varphi^{-n}(\mathbf{A}_\psi^{\dagger,p^{-n}r})$ over all $r > 0$.

Theorem 4.2.2. *Applying* **FÉt** *to any arrow in the diagram*

$$\begin{tikzcd}
 & \bigcup\varphi^{-n}(\mathbf{A}_\psi^{\dagger,p^{-n}}) \arrow[rrr] \arrow[dd] \arrow[dr] & & & \bigcup A_{\psi,n} \arrow[d] \\
 & & \widehat{\bigcup}\varphi^{-n}(\mathbf{A}_\psi^{\dagger,p^{-n}}) \arrow[r] \arrow[d] & \tilde{\mathbf{A}}_\psi^{\dagger,1} \arrow[r] \arrow[d] & A_{\psi,\infty} \\
\mathbf{A}_\psi^\dagger \arrow[r] \arrow[d] & \bigcup\varphi^{-n}(\mathbf{A}_\psi^\dagger) \arrow[r] \arrow[d] & \widehat{\bigcup}\varphi^{-n}(\mathbf{A}_\psi^\dagger) \arrow[r] \arrow[d] & \tilde{\mathbf{A}}_\psi^\dagger \arrow[d] & \\
\mathbf{A}_\psi \arrow[r] \arrow[d] & \bigcup\varphi^{-n}(\mathbf{A}_\psi) \arrow[r] & \widehat{\bigcup}\varphi^{-n}(\mathbf{A}_\psi) \arrow[r] \arrow[d] & \tilde{\mathbf{A}}_\psi \arrow[d] & \\
\mathbf{A}_\psi/(p) \arrow[rr] & & \bigcup\overline{\varphi}^{-n}(\mathbf{A}_\psi/(p)) \arrow[r] & \overline{A}_\psi &
\end{tikzcd}$$

produces a tensor equivalence.

Proof. The arrows in the bottom row of the diagram induce equivalences by [45, Theorem 3.1.14]. It is thus sufficient to link each entry of the diagram with the bottom row.

Consider first the bottom three rows of the diagram. Each of the rings

$$\mathbf{A}^{\dagger}_{\psi}, \widehat{\cup}\varphi^{-n}(\mathbf{A}^{\dagger}_{\psi}), \tilde{\mathbf{A}}^{\dagger}_{\psi}, \mathbf{A}_{\psi}, \widehat{\cup}\varphi^{-n}(\mathbf{A}_{\psi}), \tilde{\mathbf{A}}_{\psi}$$

is henselian with respect to (p); consequently, the vertical arrows from these rings induce equivalences by [45, Theorem 1.2.8]. The arrows $\cup\varphi^{-n}(\mathbf{A}^{\dagger}_{\psi}) \to \widehat{\cup}\varphi^{-n}(\mathbf{A}^{\dagger}_{\psi})$, $\cup\varphi^{-n}(\mathbf{A}_{\psi}) \to \widehat{\cup}\varphi^{-n}(\mathbf{A}_{\psi})$ thus induce functors which are essentially surjective, but also fully faithful by [45, Lemma 2.2.4(a)], and thus equivalences.

It remains to link the top two rows to the rest of the diagram. To begin with, the arrows out of $\tilde{\mathbf{A}}^{\dagger,1}_{\psi}$ induce equivalences by [45, Corollary 5.5.6], while the arrow $\cup A_{\psi,n} \to A_{\psi,\infty}$ induces an equivalence by [45, Proposition 2.6.9].

We next link $\cup\varphi^{-n}(\mathbf{A}^{\dagger,p^{-n}}_{\psi})$ to the bottom of the diagram. Any finite étale algebra U over $\mathbf{A}^{\dagger}_{\psi}$ may be defined over $\mathbf{A}^{\dagger,p^{-n}}_{\psi}$ for some n. Since any finite étale algebra over $\overline{A}_{\psi}$ is isomorphic to its $\overline{\varphi}$-pullback, we deduce (using the equivalences already established) that any finite étale algebra over $\overline{A}_{\psi}$ lifts to a finite étale algebra over $\varphi^{-n}(\mathbf{A}^{\dagger,p^{-n}}_{\psi})$ for some n. That is, the arrow $\cup\varphi^{-n}(\mathbf{A}^{\dagger,p^{-n}}_{\psi}) \to \cup\varphi^{-n}(\mathbf{A}^{\dagger}_{\psi})$ induces a functor which is essentially surjective. However, this functor is also fully faithful by the following argument. Given $U, V \in \mathbf{F\acute{E}t}(\cup\varphi^{-n}(\mathbf{A}^{\dagger,p^{-n}}_{\psi}))$, any morphism between U and V in $\mathbf{F\acute{E}t}(\cup\varphi^{-n}(\mathbf{A}^{\dagger}_{\psi}))$ gives rise (using the equivalences already established) to a morphism in $\mathbf{F\acute{E}t}(\tilde{\mathbf{A}}^{\dagger}_{\psi})$ and hence a morphism in $\mathbf{F\acute{E}t}(\tilde{\mathbf{A}}^{\dagger,1}_{\psi})$. Since U, V are finite projective modules over $\cup\varphi^{-n}(\mathbf{A}^{\dagger,p^{-n}}_{\psi})$ (see [45, Definition 1.2.1]), we also get a morphism between U and V in $\mathbf{F\acute{E}t}(R)$ for R equal to the intersection $(\cup\varphi^{-n}(\mathbf{A}^{\dagger}_{\psi})) \cap \tilde{\mathbf{A}}^{\dagger,1}_{\psi}$ within $\tilde{\mathbf{A}}^{\dagger}_{\psi}$. This intersection equals $\cup\varphi^{-n}(\mathbf{A}^{\dagger}_{\psi} \cap \tilde{\mathbf{A}}^{\dagger,p^{-n}}_{\psi}) = \cup\varphi^{-n}(\mathbf{A}^{\dagger,p^{-n}}_{\psi})$; we thus obtain full faithfulness.

We finally link $\widehat{\cup}\varphi^{-n}(\mathbf{A}^{\dagger,p^{-n}}_{\psi})$ to the bottom of the diagram. The functor induced by the arrow from this ring to $\widehat{\cup}\varphi^{-n}(\mathbf{A}^{\dagger}_{\psi})$ is essentially surjective, because we have an equivalence that factors through it. We may check full faithfulness by imitating the argument given for $\cup\varphi^{-n}(\mathbf{A}^{\dagger,p^{-n}}_{\psi})$. □

Remark 4.2.3. If ψ' is another frame obtained by making a finite étale extension of A, then we deduce from Theorem 4.2.2 (as in the proof of [45, Proposition 5.5.4]) that $\mathbf{C}_{\psi'}$ is finite étale over $\mathbf{C}_{\psi}$.

4.3 Analyticity of the Γ_N-action

As noted earlier, one key feature of the imperfect period rings which distinguishes them from their perfect counterparts is analyticity of the action of Γ_N.

Notation 4.3.1. Throughout §4.3, for n a positive integer, let U_n denote the open subgroup $(1 + p^n\mathbb{Z}_p) \ltimes p^n N_p$ of Γ_N.

Remark 4.3.2. We will make repeated use of the fact that for γ an endomorphism of a ring R, the operator $\gamma - 1$ satisfies the following analogue of the Leibniz rule: for all $x, y \in R$,

$$(\gamma - 1)(xy) = (\gamma - 1)(x)y + \gamma(x)(\gamma - 1)(y). \tag{4.3.2.1}$$

For M an R-module equipped with a semilinear γ-action, (4.3.2.1) also holds for $x \in R$, $y \in M$. This observation can be used to fold differential and difference algebra into a common framework; see [3].

Lemma 4.3.3. *Let C be a uniform affinoid algebra over a nontrivially normed analytic field, and let β be the norm on C. Let U be an open subgroup of Γ_N acting on C, and assume that $\mathcal{M}(C)$ admits a neighborhood basis each element of which is a rational subdomain stable under an open subgroup of U. Suppose also that there exists $c > 0$ such that for any positive integer n for which $U_n \subseteq U$, we have*

$$\beta(\gamma(x) - x) \leq cp^{-p^{n+1}/(p-1)}\beta(x) \qquad (x \in C, \gamma \in U_n). \tag{4.3.3.1}$$

Let D be a finite étale C-algebra (which is again a reduced affinoid algebra) and extend β to a power-multiplicative norm on D. Assume that D admits a continuous extension of the action of U. Then there exists $d > 0$ such that for any positive integer n for which $U_n \subseteq U$, we have

$$\beta(\gamma(y) - y) \leq dp^{-p^{n+1}/(p-1)}\beta(y) \qquad (y \in D, \gamma \in U_n). \tag{4.3.3.2}$$

Proof. Using (4.3.2.1), it is enough to check (4.3.3.2) for some generators of D as a finite Banach module over C (which exist by [45, Lemma 2.5.2 and Remark 2.5.3]). From this, it follows that it is enough to check (4.3.3.2) after replacing U by an open subgroup; by our hypothesis on the action of U on $\mathcal{M}(C)$, we may work locally around an arbitrary $\delta \in \mathcal{M}(C)$. (We are using here the fact that (4.3.3.1) continues to hold when C is replaced by a rational localization on which U_n acts, though possibly with a different constant c. This can be seen by writing the localization as an affinoid homomorphism.)

Since the local ring C_δ is henselian [45, Lemma 2.4.12], we may reduce to the case where the fibre of $\mathcal{M}(D) \to \mathcal{M}(C)$ above δ consists of a single point $\tilde{\delta}$. In this case, we may choose $y \in D$ whose image in $\mathcal{H}(\tilde{\delta})$ is a generator of this field over $\mathcal{H}(\delta)$. By shrinking C further, we can ensure that y is the root of a polynomial $P(T) \in C[T]$ of degree $d = [\mathcal{H}(\tilde{\delta}) : \mathcal{H}(\delta)]$. The discriminant Δ of this polynomial has nonzero image in $\mathcal{H}(\delta)$; by shrinking C further, we can ensure that Δ is a unit in C.

Write $P(T) = \sum_i P_i T^i$. Since $P(y) = \gamma(P)(\gamma(y)) = 0$, we have on one hand

$$P(\gamma(y)) = (P - \gamma(P))(\gamma(y)) = \sum_i (1 - \gamma)(P_i)\gamma(y).$$

From (4.3.3.1), we can find a constant $c > 0$ such that $\beta(\gamma(P_i) - P_i) \leq cp^{-p^{n+1}/(p-1)}$ for all positive integers n such that $U_n \subseteq U$ and all $\gamma \in U_n$.

On the other hand, we may write

$$P(\gamma(y)) = P'(y)(\gamma(y) - y) + Q(\gamma(y), y)(\gamma(y) - y)^2$$

for some polynomial $Q \in C[T_1, T_2]$. Since Δ is invertible in C, $P'(y)$ is invertible in D. Since the extension of the action of γ to D is continuous, for $\gamma \in U_n$ with n sufficiently large, we have $\beta(\gamma(y) - y) = \beta(P(\gamma(y))/P'(y)) \leq \beta(P(\gamma(y)))\beta(1/P'(y))$. We thus obtain (4.3.3.2) for this particular choice of y, and (using (4.3.2.1)) also for each power of y. Since D is generated over C by finitely many powers of y, we may deduce (4.3.3.2) in full generality. $\square$

Lemma 4.3.4. *There exists $c > 0$ such that for every positive integer n,*

$$\overline{\alpha}_\psi(\gamma(\overline{x}) - \overline{x}) \leq c p^{-p^{n+1}/(p-1)} \overline{\alpha}_\psi(\overline{x}) \qquad (\gamma \in U_n, \overline{x} \in \mathbf{A}_\psi/(p)). \tag{4.3.4.1}$$

Proof. For $A = K\{M_\sigma\}_\lambda$ for some $\lambda \in N_{\mathbb{Q}}$, using the observation that $\overline{\alpha}_\psi(\overline{\pi}^{p^n}) = p^{-p^{n+1}/(p-1)}$, it is easy to check (4.3.4.1) for $\overline{x} = 1 + \overline{\pi}$ and for $\overline{x} \in S$. The same then follows for all $\overline{x}$ thanks to (4.3.2.1).

Suppose next that ψ is a strictly rational subdomain. Set notation as in Lemma 3.1.10 and use bars to denote reduction modulo p. By writing

$$(\gamma - 1)\left(\frac{1}{\overline{g}}\right) = \frac{(1-\gamma)(\overline{g})}{\overline{g}\gamma(\overline{g})},$$

we deduce (4.3.4.1) for $\overline{x} = 1/\overline{g}$. We may then use (4.3.2.1) to check (4.3.4.1) for $\overline{x} = \overline{f}_1/\overline{g}, \dots, \overline{f}_m/\overline{g}$, and then for all $\overline{x} \in \mathbf{A}_\psi/(p)$.

Since $\mathcal{M}(\mathbf{A}_\psi/(p)) \cong \mathcal{M}(\overline{A}_\psi) \cong \mathcal{M}(A_{\psi,\infty})$ is the inverse limit of the spaces $\mathcal{M}(A_{\psi,n})$, it admits a neighborhood basis each element of which is a rational subdomain stable under an open subgroup of Γ_N. We may thus apply Lemma 4.3.3 to deduce the desired result. $\square$

Proposition 4.3.5. *There exists $c > 0$ such that for every positive integer n, there exists $r_n > 0$ such that*

$$\lambda(\overline{\alpha}^r_\psi)(\gamma(x) - x) \leq c^r p^{-p^{n+1}r/(p-1)} \lambda(\overline{\alpha}^r_\psi)(x) \qquad (\gamma \in U_n, r \in (0, r_n], x \in \mathbf{C}^r_\psi). \tag{4.3.5.1}$$

Proof. Choose $r_0 > 0$ as in Lemma 4.1.3. Given $\overline{x} \in \mathbf{A}_\psi/(p)$, lift $\overline{x}$ to $x \in \mathbf{A}^{\dagger,r_0}_\psi$ as in Lemma 4.1.3, so that $\lambda(\overline{\alpha}^{r_0}_\psi)(x - [\overline{x}]) \leq p^{-1/2}\lambda(\overline{\alpha}^{r_0}_\psi)(x)$. Write $x = \sum_{i=0}^\infty p^i[\overline{x}_i]$ and $\gamma(x) - x = \sum_{i=0}^\infty p^i[\overline{y}_i]$ with $\overline{x}_i, \overline{y}_i \in \overline{A}_\psi$. Since $\overline{x}_0 = \overline{x} \in \mathbf{A}_\psi/(p)$, for $c > 0$ as in Lemma 4.3.3, we have $\overline{\alpha}^{r_0}_\psi(\overline{y}_0) \leq c^{r_0} p^{-p^{n+1}r_0/(p-1)} \overline{\alpha}^{r_0}_\psi(\overline{x}_0)$. On the other hand, since γ is an isometry, we have $\lambda(\overline{\alpha}^{r_0}_\psi)(\gamma(x) - x) \leq \lambda(\overline{\alpha}^{r_0}_\psi)(x)$, so $\overline{\alpha}^{r_0}_\psi(\overline{y}_i) \leq \overline{\alpha}^{r_0}_\psi(\overline{x}_i)$ for $i > 0$. Now choose r_n so that

$$p^{-1/2} \leq c^{r_n} p^{-p^{n+1}r_n/(p-1)};$$

then $\lambda(\overline{\alpha}^{r_n}_\psi)(\gamma(x) - x) \leq c^{r_n} p^{-p^{n+1}r_n/(p-1)} \lambda(\overline{\alpha}^{r_n}_\psi)(x)$. From this calculation, we may deduce the general case using Corollary 4.1.4. $\square$

Corollary 4.3.6. *There exist $r_0 > 0$, $c < 1$, and a positive integer m such that*

$$\lambda(\overline{\alpha}^r_\psi)(\gamma(x) - x) \leq c^r \lambda(\overline{\alpha}^r_\psi)(x) \qquad (\gamma \in U_m, r \in (0, r_0], x \in \mathbf{C}^r_\psi). \tag{4.3.6.1}$$

Corollary 4.3.6 has the following essentially formal consequence.

Proposition 4.3.7. *For r_0 as in Corollary 4.3.6, for all $0 < s \leq r \leq r_0$, $\gamma \in \Gamma_N$, and $x \in \mathbf{C}_\psi^{[s,r]}$, the limit*

$$(d\gamma)(x) = \lim_{n \to \infty} \frac{(\gamma^{p^n(p-1)} - 1)(x)}{p^n(p-1)}$$

exists and defines a bounded derivation $d\gamma$ on $\mathbf{C}_\psi^{[s,r]}$. (The same then holds for x in $\mathbf{C}_\psi^r$ or $\mathbf{C}_\psi$.)

Proof. Take m as in Corollary 4.3.6. For $n \geq m$, write

$$\gamma^{p^n(p-1)} - 1 = \sum_{i=1}^{p^{n-m}} \binom{p^{n-m}}{i} (\gamma^{p^m(p-1)} - 1)^i;$$

since $\gamma^{p^m(p-1)} \in U_m$, we deduce that for n large,

$$\lambda(\overline{\alpha}_\psi^t)((\gamma^{p^n(p-1)} - 1)(x)) \leq p^{-(n-m)} \lambda(\overline{\alpha}_\psi^t)(x) \qquad (t \in [s,r], x \in \mathbf{C}_\psi^{[s,r]}).$$

By writing

$$\begin{aligned}\frac{\gamma^{p^{n+1}(p-1)} - 1}{p^{n+1}(p-1)} - \frac{\gamma^{p^n(p-1)} - 1}{p^n(p-1)} &= \frac{\gamma^{p^{n+1}(p-1)} - 1 - p(\gamma^{p^n(p-1)} - 1)}{p^{n+1}(p-1)} \\ &= \frac{(\gamma^{p^n(p-1)} - 1)^2}{p^{n+1}(p-1)} \sum_{i=0}^{p-2} \binom{p}{i+2} (\gamma^{p^n(p-1)} - 1)^i\end{aligned}$$

and using the fact that γ acts as an isometry, we obtain the desired convergence. □

Remark 4.3.8. To make Proposition 4.3.7 more explicit, choose a basis $T_1, \dots, T_n$ of $N^\vee$, and let $\frac{\partial}{\partial \pi}, \frac{\partial}{\partial T_1}, \dots, \frac{\partial}{\partial T_n}$ be the partial derivations on $R = (\cap_{q \in (0,1)} K\{\pi/q\})[M_\sigma]$ with respect to $\pi, T_1, \dots, T_n$. For $\gamma \in \mathbb{Z}_p^\times$, the action of $d\gamma$ on R is given by

$$d\gamma = (1 + \pi) \log(\gamma) \log(1 + \pi) \frac{\partial}{\partial \pi}.$$

For $\gamma \in N_p$, the action of $d\gamma$ on R is given by

$$d\gamma = \sum_{i=1}^n \langle \gamma, T_i \rangle T_i \log(1 + \pi) \frac{\partial}{\partial T_i}.$$

It follows that $\gamma \to d\gamma / \log(1 + \pi)$ defines an action of the Lie algebra $\mathrm{Lie}(\Gamma_N)$ on $\mathbf{C}_\psi^{[s,r]}$.

4.4 Decomposition of perfect period rings

Recall that every element of $\mathbb{F}_p((\overline{\pi}))^{\mathrm{perf}}$ can be written uniquely as a sum

$$\sum_{i\in[0,1)\cap\mathbb{Z}[p^{-1}]} c_i\overline{\pi}^i \qquad (c_i \in \mathbb{F}_p((\overline{\pi})))$$

which converges in the sense that for any $\epsilon > 0$, there are only finitely many indices i for which $|c_i| > \epsilon$. We now obtain some similar decompositions of perfect period rings in terms of their imperfect counterparts.

Definition 4.4.1. For $\mu \in (\mathbb{Z} \times N^\vee) \otimes_{\mathbb{Z}} (\mathbb{Z}[p^{-1}]/\mathbb{Z})$, let $\mathbf{A}^{\dagger,r}_\psi[\mu]$ denote the $\mathbf{A}^{\dagger,r}_\psi$-submodule of $\tilde{\mathbf{A}}^{\dagger,r}_\psi$ generated by $(1+\pi)^e s$ for all (e,s) in the inverse image of μ in $(\mathbb{Z}\times N^\vee)[p^{-1}]$. Define $\mathbf{A}_\psi[\mu], \mathbf{B}^{\dagger,r}_\psi[\mu], \mathbf{C}^{[s,r]}_\psi[\mu], \mathbf{C}^r_\psi[\mu]$ analogously.

Remark 4.4.2. If ψ is boundary-free, then $\mathbf{A}^{\dagger,r}_\psi[\mu]$ is always nonzero and is freely generated by any single element of the form $(1+\pi)^e s$. In general, one needs a finite number (possibly zero) of such generators. One can often reduce to the boundary-free case by replacing ψ with each of a sequence of rational subframes whose union is dense (using the fact that subframes of strictly rational type form a neighborhood basis in $\mathcal{M}(A)$, as in Remark 2.2.6). However, a bit of care is required: for instance, if one wishes to use this method to show that a certain map is bounded, one must check that the bound is uniform over the sequence of subframes.

Lemma 4.4.3. *The natural map*

$$\widehat{\bigoplus}_\mu \mathbf{A}_\psi[\mu]/(p) \to \tilde{\mathbf{A}}_\psi/(p)$$

is an isomorphism of Banach modules over $\mathbf{A}_\psi/(p)$ for the norm $\overline{\alpha}_\psi$ on $\tilde{\mathbf{A}}_\psi/(p)$ and the supremum norm on $\widehat{\oplus}_\mu \mathbf{A}_\psi[\mu]/(p)$ for the restriction of $\overline{\alpha}_\psi$ to each summand.

Proof. It suffices to check that the map $\oplus_\mu \mathbf{A}_\psi[\mu]/(p) \to \tilde{\mathbf{A}}_\psi/(p)$ is strict. We have an isomorphism

$$\overline{\varphi}^{-1}(\mathbf{A}_\psi/(p)) \cong \bigoplus_\nu \mathbf{A}_\psi[\nu]/(p)$$

of finite $\mathbf{A}_\psi/(p)$-modules in which ν runs over $(\mathbb{Z} \times N^\vee) \otimes_{\mathbb{Z}} (p^{-1}\mathbb{Z}/\mathbb{Z})$. Since $\mathbf{A}_\psi/(p)$ is an affinoid algebra over a nontrivially normed analytic field, this is also an isomorphism of finite Banach modules [45, Remark 2.5.3]. Consequently, there exists $c \geq 1$ such that for any $x \in \overline{\varphi}^{-1}(\mathbf{A}_\psi/(p))$, the unique decomposition $x = \sum_\nu x_\nu$ with $x_\nu \in \mathbf{A}_\psi[\nu]/(p)$ satisfies

$$\max_\nu\{\overline{\alpha}_\psi(x_\nu)\} \leq c\overline{\alpha}_\psi(x). \tag{4.4.3.1}$$

By multiplying through by some $(1+\pi)^e s$, we see that for each $\mu \in (\mathbb{Z}\times N^\vee)\otimes_{\mathbb{Z}} (\mathbb{Z}[p^{-1}]/\mathbb{Z})$, every $x \in \overline{\varphi}^{-1}(\mathbf{A}_\psi[\mu]/(p))$ has a unique representation as a sum $\sum_\nu x_\nu$ in which ν runs over elements of $(\mathbb{Z} \times N^\vee) \otimes_{\mathbb{Z}} (\mathbb{Z}[p^{-1}]/\mathbb{Z})$ with $p\nu = \mu$. Moreover, this decomposition again satisfies (4.4.3.1) for the same constant c. (This requires a reduction to the boundary-free case as in Remark 4.4.2, which is valid because the constant c can be used uniformly.)

By applying this repeatedly, we obtain for each positive integer n an isomorphism

$$\overline{\varphi}^{-n}(\mathbf{A}_\psi/(p)) \cong \bigoplus_\nu \mathbf{A}_\psi[\nu]/(p)$$

of finite Banach $\mathbf{A}_\psi/(p)$-modules in which ν runs over $(\mathbb{Z} \times N^\vee) \otimes_\mathbb{Z} (\mathbb{Z}[p^{-n}]/\mathbb{Z})$. Moreover, for any $x \in \overline{\varphi}^{-n}(\mathbf{A}_\psi/(p))$, the unique decomposition $x = \sum_\nu x_\nu$ with $x_\nu \in \mathbf{A}_\psi[\nu]/(p)$ satisfies

$$\max_\nu\{\overline{\alpha}_\psi(x_\nu)\} \leq c^{1+p^{-1}+\cdots+p^{-n-1}}\overline{\alpha}_\psi(x) \leq c^{p/(p-1)}\overline{\alpha}_\psi(x). \tag{4.4.3.2}$$

Since the final constant in (4.4.3.2) is independent of n, we may take the union over n and then complete to obtain the desired isomorphism. □

Corollary 4.4.4. *For $r_0 > 0$ as in Lemma 4.1.3, there exists $c > 0$ such that for $0 < s \leq r \leq r_0$, for $x = \sum_\mu x_\mu \in \tilde{\mathbf{C}}_\psi^{[s,r]}$, we have*

$$\sup_\mu\{\lambda(\overline{\alpha}_\psi^t)(x_\mu)\} \leq c\lambda(\overline{\alpha}_\psi^t)(x) \qquad (t \in [s,r]).$$

Consequently, the natural maps

$$\widehat{\bigoplus_\mu}\mathbf{A}_\psi^r[\mu] \to \tilde{\mathbf{A}}_\psi^r, \qquad \widehat{\bigoplus_\mu}\mathbf{B}_\psi^r[\mu] \to \tilde{\mathbf{B}}_\psi^r, \qquad \widehat{\bigoplus_\mu}\mathbf{C}_\psi^{[s,r]}[\mu] \to \tilde{\mathbf{C}}_\psi^{[s,r]}$$

are isomorphisms of Banach modules over $\mathbf{A}_\psi^r, \mathbf{B}_\psi^r, \mathbf{C}_\psi^{[s,r]}$, respectively.

Proof. Combine Lemma 4.4.3 with Corollary 4.1.4. □

Definition 4.4.5. For $r_0 > 0$ as in Lemma 4.1.3, for $0 < s \leq r \leq r_0$, we obtain from 4.4.4 a $\mathbf{C}_\psi^{[s,r]}$-linear projection $\Pi_\mu : \tilde{\mathbf{C}}_\psi^{[s,r]} \to \mathbf{C}_\psi^{[s,r]}$ by picking out one term in the decomposition $\tilde{\mathbf{C}}_\psi^{[s,r]} \cong \widehat{\oplus}_\mu \mathbf{C}_\psi^{[s,r]}[\mu]$. In case ψ is boundary-free, we can compose Π_μ with an isomorphism $\mathbf{C}_\psi^{[s,r]}[\mu] \cong \mathbf{C}_\psi^{[s,r]}$ (see Remark 4.4.2) to obtain a projection $\tilde{\mathbf{C}}_\psi^{[s,r]} \to \mathbf{C}_\psi^{[s,r]}$. See [42, §3.5] for a similar construction.

4.5 φ-modules over imperfect period rings

We now introduce imperfect period rings into the study of local systems, starting with a study of φ-modules parallel to the study made for perfect period rings in [45, §6–7]. We do not complete the analogy, however, because for (φ, Γ_N)-modules we will be able to descend results from the perfect case more easily (see Theorem 4.9.1).

Hypothesis 4.5.1. Throughout §4.5, take $r_0 > 0$ as in Lemma 4.1.3 and Lemma 4.1.5. let a be a positive integer, and put $q = p^a$.

We start with the analogue of [45, Definition 6.1.1].

Definition 4.5.2. A φ^a*-module* over an imperfect period ring is a finite projective module equipped with a semilinear φ^a-action (i.e., an isomorphism with its pullback by φ^a). These form an exact (but not abelian) category in which the morphisms are φ^a-equivariant morphisms of underlying modules. As in [45, Definition 1.5.3], for M a φ^a-module, we write

$$H^0_{\varphi^a}(M) = \ker(\varphi^a - 1, M), \qquad H^1_{\varphi^a}(M) = \operatorname{coker}(\varphi^a - 1, M),$$

and $H^i_{\varphi^a}(M) = 0$ for $i \geq 2$.

For $* \in \{\mathbf{A}^\dagger, \mathbf{B}^\dagger, \mathbf{C}\}$ and M a φ^a-module over $*_\psi$, for $r > 0$ sufficiently small, M can be realized as the base extension of a finite projective module M_r over $*^r_\psi$ equipped with an isomorphism $\varphi^* M_r \cong M_r \otimes_{*^r_\psi} *^{r/q}_\psi$. We call M_r a *model* of M over $*^r_\psi$.

Lemma 4.5.3. *Take R to be one of $\mathbf{A}_\psi, \widehat{\cup}\varphi^{-n}(\mathbf{A}_\psi), \tilde{\mathbf{A}}_\psi$. Let M be a finite projective module over $R/(p^m)$ for some positive integer m equipped with an semilinear φ^a-action. Then there exists a faithfully finite étale R-algebra U such that $M \otimes_{R/(p^m)} U/(p^m)$ admits a basis fixed by φ^a. More precisely, if $l < m$ is another positive integer and $M \otimes_{R/(p^m)} R/(p^l)$ admits a φ^a-fixed basis, then U can be chosen so that this basis lifts to a φ^a-fixed basis of $M \otimes_{R/(p^m)} U/(p^m)$.*

Proof. As in [45, Lemma 3.2.6]. ☐

Corollary 4.5.4. *The base change functors among the categories of φ^a-modules over the rings*

$$\mathbf{A}_\psi \to \bigcup \varphi^{-n}(\mathbf{A}_\psi) \to \widehat{\bigcup} \varphi^{-n}(\mathbf{A}_\psi) \to \tilde{\mathbf{A}}_\psi$$

are tensor equivalences.

Proof. It is enough to check that for each ring R in the diagram, the base change functor from R to $\tilde{\mathbf{A}}_\psi$ is an equivalence of categories. To prove full faithfulness, it suffices to prove that for any φ^a-module M over R, any $\mathbf{v} \in M \otimes_R \tilde{\mathbf{A}}_\psi$ fixed by φ^a must belong to M. For this, we may omit the case $R = \cup\varphi^{-n}(\mathbf{A}_\psi)$ by reducing it to the case $R = \mathbf{A}_\psi$; we may thus assume that R is p-adically complete. Apply Lemma 4.5.3 to construct a R-algebra U which is the p-adic completed direct limit of faithfully finite étale R-subalgebras, such that $M \otimes_R U$ admits a φ^a-fixed basis. Put $\tilde{U} = U \widehat{\otimes}_R \tilde{\mathbf{A}}_\psi$, so that we may view $\mathbf{v}$ as an element of $M \otimes_R \tilde{U}$. By writing $\mathbf{v}$ in terms of a φ^a-fixed basis of $M \otimes_R U$ and noting that $U^{\varphi^a} = \tilde{U}^{\varphi^a}$, we deduce that $\mathbf{v} \in M \otimes_R U$. Since within $\tilde{U}$ we have $\tilde{\mathbf{A}}_\psi \cap U = R$, we deduce that $\mathbf{v} \in M$ as desired.

To prove essential surjectivity, it is enough to factor the functor from $\mathbb{Z}_p$-$\mathbf{Loc}(\overline{A}_\psi)$ to φ^a-modules over $\tilde{\mathbf{A}}_\psi$ through the category of φ^a-modules over $\mathbf{A}_\psi$. To do this, use Lemma 4.5.3 to imitate the construction in the proof of [45, Theorem 8.1.2]. ☐

As in [45], it is occasionally useful to consider a variant defined in more geometric terms.

Definition 4.5.5. For $r \in (0, r_0]$, let $X_{\psi,r}$ be the quasi-Stein space (Definition 1.3.1) obtained by taking the union of the strictly affinoid (by Lemma 4.1.5) spaces $\mathcal{M}(\mathbf{C}_\psi^{[s,r]})$ over all $s > 0$. Note that φ^a induces a map $X_{\psi,r/q} \to X_r$; a *φ^a-bundle* over $\mathbf{C}_\psi^r$ is a vector bundle $\mathcal{E}$ over $X_{\psi,r}$ equipped with an isomorphism $(\varphi^a)^*\mathcal{E} \cong \mathcal{E}$ of vector bundles over $X_{\psi,r/q}$. By taking the categorical direct limit [45, Remark 1.2.9] over all $r \in (0, r_0]$, we obtain the category of *φ^a-bundles* over $\mathbf{C}_\psi$.

For $s \in (0, r/q]$, a *φ^a-module* over $\mathbf{C}_\psi^{[s,r]}$ is a finite locally free module M over $\mathbf{C}_\psi^{[s,r]}$ equipped with an isomorphism $(\varphi^a)^*M \otimes_{\mathbf{C}_\psi^{[s/q,r/q]}} \mathbf{C}_\psi^{[s,r/q]} \cong M \otimes_{\mathbf{C}_\psi^{[s,r]}} \mathbf{C}_\psi^{[s,r/q]}$ of modules over $\mathbf{C}_\psi^{[s,r/q]}$. There is a functor from φ^a-bundles over $\mathbf{C}_\psi^r$ to φ^a-modules over $\mathbf{C}_\psi^{[s,r]}$ obtained by taking sections of the bundle over the affinoid space $\mathcal{M}(\mathbf{C}_\psi^{[s,r]})$; as in [45, Lemma 6.1.3], this functor is a tensor equivalence thanks to Kiehl's theorem [45, Theorem 2.5.13(b)].

Lemma 4.5.6. *Let $\mathcal{E}$ be a φ^a-bundle over $\mathbf{C}_\psi^r$ for some $r \in (0, r_0]$. Let M_r be the module of global sections of $\mathcal{E}$. Then M_r is a finite projective module over $\mathbf{C}_\psi^r$.*

Proof. Let $\tilde{\mathcal{E}}$ be the corresponding φ^a-bundle over $\tilde{\mathbf{C}}_\psi^r$, and let $\tilde{M}_r$ be the module of global sections of $\tilde{\mathcal{E}}$. By Theorem 1.3.2, there exists a finite subset S of M_r which generates the module $M_r \otimes_{\mathbf{C}_\psi^r} \mathbf{C}_\psi^{[r/q,r]}$ over $\mathbf{C}_\psi^{[r/q,r]}$, and hence also the module $M_r \otimes_{\mathbf{C}_\psi^r} \tilde{\mathbf{C}}_\psi^{[r/q,r]}$ over $\tilde{\mathbf{C}}_\psi^{[r/q,r]}$. By applying powers of φ^a and then invoking [45, Lemma 6.1.6], we see that S also generates $\tilde{M}_r$ as a module over $\tilde{\mathbf{C}}_\psi^r$. It follows that the natural map $M_r \otimes_{\mathbf{C}_\psi^r} \tilde{\mathbf{C}}_\psi^r \to \tilde{M}_r$ is surjective; this map may also be seen to be injective by approximating ψ with boundary-free subframes (Remark 4.4.2) and then tensoring with the projections $\Pi_\mu : \tilde{\mathbf{C}}_\psi^r \to \mathbf{C}_\psi^r[\mu] \cong \mathbf{C}_\psi^r$ provided by Definition 4.4.5.

By [45, Theorem 6.4.2], $\tilde{M}_r$ is a direct summand of a finite free module over $\tilde{\mathbf{C}}_\psi^r$. By tensoring this presentation with one of the projections $\tilde{\mathbf{C}}_\psi^r \to \mathbf{C}_\psi^r$, we deduce that M_r is a direct summand of a finite free module over $\mathbf{C}_\psi^r$, as desired. □

Theorem 4.5.7. *The natural functor from φ^a-modules over $\mathbf{C}_\psi$ to φ^a-bundles over $\mathbf{C}_\psi$ is a tensor equivalence.*

Proof. Full faithfulness follows from Corollary 4.1.6, while essential surjectivity follows from Lemma 4.5.6. □

Corollary 4.5.8. *The category of φ^a-modules over $\mathbf{C}_\psi$ admits glueing for covering families of rational subframes.*

Proof. This follows from Theorem 4.5.7 together with Kiehl's theorem [45, Theorem 2.5.13(b)]. □

Remark 4.5.9. When A is a finite extension of K, one can obtain Theorem 4.5.7 without using the φ-action, using Lazard's theorem that any vector bundle on an annulus over a spherically complete field is freely generated by global sections [49]. It is unclear whether such an argument can be made more generally.

We have the following analogue of [45, Proposition 6.4.5].

Proposition 4.5.10. *Let M be a φ^a-module over $\mathbf{C}_\psi$, and let M_r be a model of M over $\mathbf{C}^r_\psi$ for some $r \in (0, r_0]$. For r', s with $0 \leq r' \leq r$ and $s \in (0, r'/q]$, put $M_{[s,r']} = M_r \otimes_{\mathbf{C}^r_\psi} \mathbf{C}^{[s,r']}_\psi$. Then the vertical arrows in the diagram*

$$\begin{tikzcd}
0 \arrow[r] & M_r \arrow[r, "\varphi^a - 1"] \arrow[d] & M_{r/q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{[s,r']} \arrow[r, "\varphi^a - 1"] & M_{[s,r'/q]} \arrow[r] & 0
\end{tikzcd}$$

induce an isomorphism on the cohomology of the horizontal complexes. Consequently, if we write $H^i_{\varphi^a}(M_r), H^i_{\varphi^a}(M_{[s,r']})$ for the cohomology of the horizontal complexes, we obtain natural isomorphisms $H^i_{\varphi^a}(M) \cong H^i_{\varphi^a}(M_r) \cong H^i_{\varphi^a}(M_{[s,r']})$.

Proof. Identify the kernel and cokernel with extension groups and apply Lemma 4.5.6. $\square$

Definition 4.5.11. We define *rank*, *degree*, and *slope* for φ^a-modules over an imperfect period ring by analogy with the perfect case (see [45, Definition 7.2.1]). In fact, these invariants can all be computed by first extending scalars to the corresponding perfect ring.

Define the *pure locus* (resp. the *étale locus*) of M by base extension to the corresponding perfect period ring. These are open subsets of $\mathcal{M}(A)$ by Remark 3.4.6.

We define the pure and étale conditions over imperfect period rings by following [45, §7.3].

Definition 4.5.12. Choose R from among $\mathbf{B}_\psi, \mathbf{B}^\dagger_\psi, \mathbf{C}_\psi$, then define R_0 to be $\mathbf{A}_\psi, \mathbf{A}^\dagger_\psi, \mathbf{A}^\dagger_\psi$, respectively. Fix integers c, d with d a positive multiple of a. Let M be a φ^a-module over R. A *(c,d)-pure model* for M is a finite R_0-submodule M_0 of M for which the map $M_0 \otimes_{R_0} R \to M$ is an isomorphism, and the action of φ^a on M induces an isomorphism $(p^c \varphi^d)^* M_0 \cong M_0$. We say the pure model is *free* or *locally free* if the underlying R_0-module M_0 has that property.

For $\gamma \in \mathcal{M}(A)$, a *(free, locally free) local (c,d)-pure model* for M at γ consists of a rational subframe ψ' encircling γ and a (free, locally free) (c,d)-pure model of $M \otimes_R R'$ for R' the analogue of R with ψ replaced by ψ'. We write *étale model* as shorthand for $(0,1)$-*pure model*, and likewise in the presence of modifiers. (Note that we must assume that ψ' again satisfies Hypothesis 4.0.1, but this is harmless in practice because such subframes form a neighborhood basis of γ in $\mathcal{M}(A)$, as in Remark 2.2.6.)

It can be shown (by arguing as in [45, Definition 8.1.5]) that the existence of a (c,d)-pure model depends only on the ratio c/d. If M admits a local (c,d)-pure model at γ, we say M is *pure* or *pure of slope c/d* at γ. If M is pure (resp. pure of slope s) at all $\gamma \in \mathcal{M}(A)$, we simply say that M is *pure* (resp. *pure of slope s*). In these definitions, we write *étale* as shorthand for *pure of slope* 0.

Lemma 4.5.13. *Keep notation as in Definition 4.5.12.*

(a) *The φ^a-module M admits a locally free local (c,d)-pure model at β if and only if it admits a free local (c,d)-pure model at β.*

(b) *If M is a φ^a-module over $\mathbf{B}^\dagger_\psi$, then M admits a free local (c,d)-pure model at β if and only if $M \otimes_{\mathbf{B}^\dagger_\psi} \mathbf{B}_\psi$ does.*

Proof. As in [45, Lemma 7.3.2]. □

Proposition 4.5.14. (a) *The base change functor from φ^a-modules over $\mathbf{A}^\dagger_\psi$ to φ^a-modules over $\mathbf{A}_\psi$ is fully faithful.*

(b) *The base change functor from globally étale φ^a-modules over $\mathbf{B}^\dagger_\psi$ to φ^a-modules over $\mathbf{C}_\psi$ is fully faithful.*

Proof. As in [45, Remark 4.3.4], part (a) reduces to checking that any φ^a-invariant element $\mathbf{v}$ of $M \otimes_{\mathbf{A}^\dagger_\psi} \mathbf{A}_\psi$ belongs to M itself. By extending scalars to $\tilde{\mathbf{A}}_\psi$ and applying [45, Theorem 8.1.2], we deduce that $\mathbf{v}$ belongs to $M \otimes_{\mathbf{A}^\dagger_\psi} \tilde{\mathbf{A}}^\dagger_\psi$. Since M is projective and we have $\mathbf{A}_\psi \cap \tilde{\mathbf{A}}^\dagger_\psi = \mathbf{A}^\dagger_\psi$ within $\tilde{\mathbf{A}}_\psi$, we deduce that $\mathbf{v} \in M$ as desired.

Similarly, part (b) reduces to checking that any φ^a-invariant element $\mathbf{v}$ of $M \otimes_{\mathbf{B}^\dagger_\psi} \mathbf{C}_\psi$ belongs to M itself. By extending scalars to $\tilde{\mathbf{B}}^\dagger_\psi$ and applying [45, Theorem 8.1.4], we deduce that $\mathbf{v}$ belongs to $M \otimes_{\mathbf{B}^\dagger_\psi} \tilde{\mathbf{B}}^\dagger_\psi$. Since $\mathbf{B}_\psi \cap \tilde{\mathbf{B}}^\dagger_\psi = \mathbf{B}^\dagger_\psi$ within $\tilde{\mathbf{B}}_\psi$, we deduce that $\mathbf{v} \in M$ as desired. □

4.6 (φ, Γ_N)-modules

We now introduce (φ, Γ_N)-modules over imperfect period rings.

Definition 4.6.1. For d a positive integer, a (φ^d, Γ_N)-*module* over an imperfect period ring is (as expected) a φ^d-module equipped with a compatible action of Γ_N which is effective and continuous for all available topologies. (Effectivity here means that for each $s \in M_\sigma$, the subgroup of N_p fixing s acts trivially on the module modulo s.) We say such an object is *(globally) pure/étale* if the same is true of the underlying φ^d-module, with no regard for the action of Γ_N. Define the cohomology groups $H^i_{\varphi^d,\Gamma}(M)$ as in Definition 3.4.7.

Remark 4.6.2. Because ψ is now forced to be étale by hypothesis, Remark 3.2.4 implies that one may omit the hypothesis of continuity of the Γ_N-action in the definition of a (φ^d, Γ_N)-module over $\mathbf{A}_\psi$ or $\mathbf{A}^\dagger_\psi$. Similarly, by Proposition 3.5.7, one may replace continuity with boundedness in the definition of a (φ^d, Γ_N)-module over $\mathbf{C}_\psi$.

Remark 4.6.3. Over an imperfect period ring, continuity of the action of Γ_N on a (φ^d, Γ_N)-module implies that the operator norm of $\gamma - 1$ tends to 0 as $\gamma \in \Gamma_N$ tends to 1. Namely, this holds for the trivial (φ^d, Γ_N)-module by Corollary 4.3.6, and extends to the general case by (4.3.2.1).

Once $\gamma - 1$ acts with sufficiently small operator norm (e.g., less than p^{-1}), the identity

$$\gamma^p - 1 = (1 + (\gamma - 1))^p - 1 = (\gamma - 1)^p + p \sum_{i=1}^{p-1} \binom{p-1}{i-1} \frac{(\gamma - 1)^i}{i}$$

may be used repeatedly to deduce that the action of Γ_N is analytic.

4.7 Deperfection

We make some calculations with the aim of descending (φ, Γ_N)-modules from $\tilde{\mathbf{C}}_\psi$ to $\mathbf{C}_\psi$. These are in the spirit of the work of Sen [64] as reinterpreted by Cherbonnier and Colmez [18].

Hypothesis 4.7.1. Throughout §4.7, choose $r_0 > 0$ as in Lemma 4.1.3. Fix an identification $N \cong \mathbb{Z}^n$, let H be the subgroup $(1 + p^2\mathbb{Z}_p) \ltimes p^2\mathbb{Z}_p^n$ of Γ_N, put $\gamma_0 = 1 + p^2 \in 1 + p^2\mathbb{Z}_p$, and let $\gamma_1, \dots, \gamma_n$ be the standard generators of $p^2 N_p \cong p^2\mathbb{Z}_p^n$. We may view $\gamma_0, \dots, \gamma_n$ as topological generators of H, even when $p = 2$.

We construct some subrings intermediate between the imperfect and perfect period rings.

Definition 4.7.2. Take $r \in (0, r_0]$ and $s \in (0, r]$. From Corollary 4.4.4, we have isomorphisms

$$\overline{A}_\psi = \tilde{\mathbf{A}}_\psi/(p) \cong \widehat{\bigoplus_\mu} \mathbf{A}_\psi[\mu]/(p), \qquad \tilde{\mathbf{C}}_\psi^{[s,r]} \cong \widehat{\bigoplus_\mu} \mathbf{C}_\psi^{[s,r]}[\mu]$$

of Banach modules over $\mathbf{A}_\psi, \mathbf{C}_\psi^{[s,r]}$, respectively. Regroup terms to obtain

$$\overline{A}_\psi \cong \mathbf{A}_\psi/(p) \oplus \overline{N}_{0,\psi} \oplus \cdots \oplus \overline{N}_{n,\psi}, \qquad \tilde{\mathbf{C}}_\psi^{[s,r]} \cong \mathbf{C}_\psi^{[s,r]} \oplus N_{[s,r],0,\psi} \oplus \cdots \oplus N_{[s,r],n,\psi}$$

with $\overline{N}_{0,\psi}, N_{[s,r],0,\psi}$ accounting for those μ with nonzero image in $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ and $\overline{N}_{i,\psi}, N_{[s,r],i,\psi}$ for $i > 0$ accounting for those μ for which $\mu \in N^\vee \otimes_{\mathbb{Z}} (\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})$, $\langle \gamma_i/p^2, \mu \rangle \neq 0$, and $\langle \gamma_j/p^2, \mu \rangle = 0$ for $j = i+1, \dots, n$. Then put

$$\overline{A}_\psi^i = \mathbf{A}_\psi/(p) \oplus \overline{N}_{0,\psi} \oplus \cdots \oplus \overline{N}_{i,\psi}; \qquad \mathbf{C}_\psi^{[s,r],i} = \mathbf{C}_\psi^{[s,r]} \oplus N_{[s,r],0,\psi} \oplus \cdots \oplus N_{[s,r],i,\psi};$$

these are complete Γ_N-stable subrings of $\overline{A}_\psi, \tilde{\mathbf{C}}_\psi^{[s,r]}$. Note that $\overline{A}_\psi^n = \overline{A}_\psi, \mathbf{C}_\psi^{[s,r],n} = \tilde{\mathbf{C}}_\psi^{[s,r]}$; by convention, we also put $\overline{A}_\psi^{-1} = \mathbf{A}_\psi/(p), \mathbf{C}_\psi^{[s,r],-1} = \mathbf{C}_\psi^{[s,r]}$.

Using these intermediate rings as stepping stones, we descend (φ, Γ_N)-modules from $\tilde{\mathbf{C}}_\psi$ to $\mathbf{C}_\psi$ as follows.

Lemma 4.7.3. *For $i = 0, \dots, n$, for any nonnegative integer h, $\gamma_i^{p^h} - 1$ is bijective on $\overline{N}_{i,\psi}$ with bounded inverse.*

Proof. By Lemma 4.4.3, it is sufficient to check that for each μ which contributes to $\overline{N}_{i,\psi}$, $\gamma_i - 1$ is bijective on $\mathbf{A}_\psi[\mu]/(p)$ and its inverse is bounded uniformly in μ. Take $c > 0$ as in Lemma 4.3.4, then choose a positive integer h for which $c < p^{p^{h+2}}$. By (4.3.4.1), we have

$$\overline{\alpha}_\psi(\gamma_i^{p^h}(\overline{x}) - \overline{x}) \leq cp^{-p^{h+3}/(p-1)}\overline{\alpha}_\psi(\overline{x}) \qquad (\overline{x} \in \mathbf{A}_\psi/(p)). \tag{4.7.3.1}$$

On the other hand, if we choose $(e, s) \in (\mathbb{Z} \times N^\vee)[\frac{1}{p}]$ lifting μ and put $\overline{y} = (1 + \overline{\pi})^e s$, then $\gamma_i^{p^h}(\overline{y}) - \overline{y} = (1 + \overline{\pi})^t \overline{y}$ for some $t \in \mathbb{Z}[p^{-1}]$ of p-adic valuation at most $h + 1$. Therefore

$$\overline{\alpha}_\psi(\gamma_i^{p^h}(\overline{y}) - \overline{y}) \geq p^{-p^{h+2}/(p-1)}\overline{\alpha}_\psi(\overline{y}). \tag{4.7.3.2}$$

Note that $cp^{-p^{h+3}/(p-1)} < p^{-p^{h+2}/(p-1)}$ by our choice of c. By combining (4.7.3.1) and (4.7.3.2) using (4.3.2.1) (and reducing to the boundary-free case as in Remark 4.4.2), we deduce that

$$\overline{\alpha}_\psi(\gamma_i^{p^h}(\overline{x}) - \overline{x}) = p^{-p^{h+2}/(p-1)}\overline{\alpha}_\psi(\overline{x}) \qquad (\overline{x} \in \mathbf{A}_\psi[\mu]/(p)). \tag{4.7.3.3}$$

From (4.7.3.3) and the expression

$$\gamma_i^{p^h} - 1 = (\gamma_i - 1)(1 + \gamma_i + \cdots + \gamma_i^{p^h - 1}),$$

we deduce the desired result. ☐

Corollary 4.7.4. *For $i = 0, \dots, n$ and h a nonnegative integer, the map*

$$\overline{A}_\psi^i \cong \overline{A}_\psi^{i-1} \oplus \overline{N}_{i,\psi} \xrightarrow{1 \oplus (\gamma_i^{p^h} - 1)} \overline{A}_\psi^{i-1} \oplus \overline{N}_{i,\psi} \cong \overline{A}_\psi^i$$

is bijective with bounded inverse.

Proof. Combine Lemma 4.4.3 and Lemma 4.7.3. ☐

Corollary 4.7.5. *For $i = 0, \dots, n$ and h a nonnegative integer, there exists $c \geq 1$ such that for $0 < s \leq r \leq r_0$, every $x \in \mathbf{C}_\psi^{[s,r],i}$ has a unique representation as $y + (\gamma_i^{p^h} - 1)z$ with $y \in \mathbf{C}_\psi^{[s,r],i}$, $z \in N_{[s,r],i,\psi}$, and this representation satisfies*

$$\lambda(\overline{\alpha}_\psi^t)(y), \lambda(\overline{\alpha}_\psi^t)(z) \leq c^t \lambda(\overline{\alpha}_\psi^t)(x) \qquad (t \in [s, r]).$$

Proof. Combine Corollary 4.7.4 with Corollary 4.1.4. ☐

Lemma 4.7.6. *Choose $i \in \{0, \dots, n\}$, h a nonnegative integer, $r \in (0, r_0]$, and $s \in (0, r]$. Let G_i be a square matrix over $\mathbf{C}_\psi^{[s,r],i}$ such that $\lambda(\overline{\alpha}_\psi^t)(G_i - 1) < 1$ for all $t \in [s, r]$. Then for any sufficiently large nonnegative integer a, there exists a square matrix U over $\varphi^{-a}(\mathbf{C}_\psi^{[sp^{-a}, rp^{-a}],i})$ of the same size such that $\lambda(\overline{\alpha}_\psi^t)(U - 1) < 1$ for all $t \in [s, r]$ and $U^{-1} G_i \gamma_i^{p^h}(U)$ has entries in $\varphi^{-a}(\mathbf{C}_\psi^{[sp^{-a}, rp^{-a}],i-1})$.*

Proof. Take $c \geq 1$ as in Corollary 4.7.5, and put

$$\epsilon = \sup\{\lambda(\overline{\alpha}^t_\psi)(G_i - 1)^{1/3} : t \in [s, r]\}.$$

We prove the claim for any a for which $\epsilon c^{rp^{-a}} < 1$, by defining a sequence of invertible matrices $U_0, U_1, \dots$ over $\varphi^{-a}(\mathbf{C}^{[sp^{-a}, rp^{-a}], i}_\psi)$ for which that the representation of $G_{i,l} = U_l^{-1} G_i \gamma_i^{p^h}(U_l)$ as $1 + X_l + (\gamma_i^{p^h} - 1)(Y_l)$ with X_l having entries in $\varphi^{-a}(\mathbf{C}^{[sp^{-a}, rp^{-a}], i-1}_\psi)$ and Y_l having entries in $\varphi^{-a}(N_{[sp^{-a}, rp^{-a}], i, \psi})$ satisfies

$$\lambda(\overline{\alpha}^t_\psi)(X_l) \leq \epsilon^2, \qquad \lambda(\overline{\alpha}^t_\psi)(Y_l) \leq \epsilon^{l+2} \qquad (t \in [s, r]).$$

The product $U_0 U_1 \cdots$ will then converges under $\lambda(\overline{\alpha}^t_\psi)$ for $t \in [s, r]$ to a matrix U of the desired form.

To begin with, put $U_0 = 1$ and apply Corollary 4.7.5 to construct X_0 and Y_0 of the desired form with $\lambda(\overline{\alpha}^t_\psi)(X_0), \lambda(\overline{\alpha}^t_\psi)(Y_0) \leq c^{tp^{-a}} \lambda(\overline{\alpha}^r_\psi)(G_i - 1) \leq \epsilon^2$ for $t \in [s, r]$. Given U_l, X_l, Y_l, set $U_{l+1} = U_l(1 - Y_l)$, so that

$$G_{i,l+1} = (1 - Y_l)^{-1}(1 + X_l + (\gamma_i^{p^h} - 1)(Y_l))(1 - \gamma_i^{p^h}(Y_l)) = 1 + X_l + Y_l X_l - X_l \gamma_i^{p^h}(Y_l) + E_l$$

for some matrix E_l with $\lambda(\overline{\alpha}^t_\psi)(G_{i,l+1} - E_l) \leq \epsilon^{2(l+2)}$ for $t \in [s, r]$. Note that $Y_l X_l - X_l \gamma_i^{p^h}(Y_l)$ has entries in $\varphi^{-a}(N_{[sp^{-a}, rp^{-a}], i, \psi})$ and $\lambda(\overline{\alpha}^t_\psi)(Y_l X_l - X_l \gamma_i^{p^h}(Y_l)) \leq \epsilon^{l+4}$ for $t \in [s, r]$. By Corollary 4.7.5, we can split $E_l = A_l + (\gamma_i^{p^h} - 1)(B_l)$ with A_l having entries in $\varphi^{-a}(\mathbf{C}^{[sp^{-a}, rp^{-a}], i-1}_\psi)$, B_l having entries in $\varphi^{-a}(N_{[sp^{-a}, rp^{-a}], i, \psi})$, and $\lambda(\overline{\alpha}^t_\psi)(A_l), \lambda(\overline{\alpha}^t_\psi)(B_l) \leq c^{tp^{-a}} \lambda(\overline{\alpha}^t_\psi)(E_l) \leq c^{tp^{-a}} \epsilon^{l+4} \leq \epsilon^{l+3}$. We may then take $X_{l+1} = X_l + A_l$ and $Y_{l+1} = B_l$. □

Lemma 4.7.7. *Let d be a positive integer and put $q = p^d$. Let M be a (φ^d, Γ_N)-module over $\tilde{\mathbf{C}}^\dagger_\psi$. Let M_r be a model of M over $\tilde{\mathbf{C}}^r_\psi$ for some $r \in (0, r_0]$. Suppose that $M_r \otimes_{\tilde{\mathbf{C}}^r_\psi} \tilde{\mathbf{C}}^{[r/q, r]}_\psi$ admits a basis $\mathbf{e}_1, \dots, \mathbf{e}_m$, and let $F, G_0, \dots, G_n$ be the matrices via which $\varphi, \gamma_0, \dots, \gamma_n$ act on the given basis. Suppose further that for some $i \in \{0, \dots, n\}$ and some nonnegative integer h, F has entries in $\mathbf{C}^{[r/q, r/q], i}_\psi$, $G_0, \dots, G_n$ have entries in $\mathbf{C}^{[r/q, r], i}_\psi$, and $H_h = G_i \gamma_i(G_i) \cdots \gamma_i^{p^h - 1}(G_i)$ has entries in $\mathbf{C}^{[r/q, r], i-1}_\psi$. Then F has entries in $\mathbf{C}^{[r/q, r/q], i-1}_\psi$ and $G_0, \dots, G_n$ all have entries in $\mathbf{C}^{[r/q, r], i-1}_\psi$.*

Proof. Since Γ_N acts continuously on M, we may increase h to ensure that $\lambda(\overline{\alpha}^s_\psi)(H_h - 1) < p^{-1}$ for $s \in [r/q, r]$. There is no harm in applying φ^d to everything in order to replace r by r/q; by so doing, thanks to Corollary 4.7.5 we can ensure that

$$\lambda(\overline{\alpha}^s_\psi)(x) \leq p \lambda(\overline{\alpha}^s_\psi)(\gamma_i^{p^h}(x) - x) \qquad (x \in N_{[r/q, r], i, \psi}, s \in [r/q, r]).$$

Since γ_i commutes with φ and γ_j, we have

$$H_h^{-1} F \varphi(H_h) = \gamma_i^{p^h}(F), \qquad H_h^{-1} G_j \gamma_j(H_h) = \gamma_i^{p^h}(G_j).$$

Write $F = F_1 + F_2$, $G_j = G_{j,1} + G_{j,2}$ with F_1 having entries in $\mathbf{C}_\psi^{[r/q,r/q],i-1}$, F_2 having entries in $N_{[r/q,r/q],i,\psi}$, $G_{j,1}$ having entries in $\mathbf{C}_\psi^{[r/q,r],i-1}$, and $G_{j,2}$ having entries in $N_{[r/q,r],i,\psi}$. Then

$$H_h^{-1} F_2 \varphi(H_h) - F_2 = (\gamma_i^{p^h} - 1)(F_2), \qquad H_h^{-1} G_{j,2} \gamma_j(H_h) - G_{j,2} = (\gamma_i^{p^h} - 1)(G_{j,2}).$$

Suppose that $F_2 \neq 0$. Since $\lambda(\overline{\alpha}_\psi^s)(H_h - 1) < p^{-1}$ for $s \in [r/q, r]$, we have $\lambda(\overline{\alpha}_\psi^{r/q})(H_h^{-1} F_2 \varphi(H_h) - F_2) < p^{-1} \lambda(\overline{\alpha}_\psi^{r/q})(F_2)$. On the other hand, $\lambda(\overline{\alpha}_\psi^{r/q})(\gamma_i^{p^h} - 1)(F_2) \geq p^{-1} \lambda(\overline{\alpha}_\psi^{r/q})(F_2)$, a contradiction. We deduce that $F_2 = 0$; a similar argument shows that $G_{j,2} = 0$. This proves the claim. □

Proposition 4.7.8. *Let d be a positive integer and put $q = p^d$. Let M be a (φ^d, Γ_N)-module over $\tilde{\mathbf{C}}_\psi^\dagger$. Let M_r be a model of M over $\tilde{\mathbf{C}}_\psi^r$ for some $r \in (0, r_0]$. Suppose that $M_{[r/q,r]} = M_r \otimes_{\tilde{\mathbf{C}}_\psi^r} \tilde{\mathbf{C}}_\psi^{[r/q,r]}$ is a free module. Then for a suitable choice of r, $M_{[r/q,r]}$ admits a basis on which φ acts via a matrix over $\mathbf{C}_\psi^{[r/q,r/q]}$ and $\gamma_0, \dots, \gamma_n$ act via matrices over $\mathbf{C}_\psi^{[r/q,r]}$.*

Proof. We check that for $i = n, \dots, -1$, we can find a basis of M on which φ acts via a matrix over $\mathbf{C}_\psi^{[r/q,r/q],i}$ and $\gamma_0, \dots, \gamma_n$ act via matrices over $\mathbf{C}_\psi^{[r/q,r],i}$. This holds for $i = n$ because $\mathbf{C}_\psi^{[r/q,r],n} = \tilde{\mathbf{C}}_\psi^{[r/q,r]}$. Given the claim for some i, by Lemma 4.7.6 and continuity of the action of γ, we can change basis so that for some nonnegative integer h, $\gamma_i^{p^h}$ acts via a matrix over $\mathbf{C}_\psi^{[r/q,r],i-1}$ for some $r > 0$. (More precisely, after applying Lemma 4.7.6, apply φ^a and replace r with rq^{-a}.) On this new basis, φ acts via a matrix over $\mathbf{C}_\psi^{[r/q,r/q],i-1}$ and $\gamma_0, \dots, \gamma_n$ act via matrices over $\mathbf{C}_\psi^{[r/q,r],i-1}$ by Lemma 4.7.7. This completes the induction; the case $i = -1$ yields the desired result. □

Similar but simpler arguments allow us also to compare cohomology over $\mathbf{C}_\psi$ and $\tilde{\mathbf{C}}_\psi$.

Lemma 4.7.9. *Let d be a positive integer and put $q = p^d$. Let M be a (φ^d, Γ_N)-module over $\mathbf{C}_\psi$. Let M_r be a model of M over $\tilde{\mathbf{C}}_\psi^r$ for some $r \in (0, r_0]$. Then for $i = 0, \dots, n$, there exists a positive integer a such that $\gamma_i - 1$ acts bijectively on $M_r \otimes_{\mathbf{C}_\psi^r} \varphi^{-ad}(N_{[r/q^{a+1}, r/q^a],i,\psi})$.*

Proof. Choose a finite Banach norm on $M_{[r/q,r]} = M_r \otimes_{\mathbf{C}_\psi^r} \mathbf{C}_\psi^{[r/q,r]}$. By continuity of the Γ_N-action, for any $\epsilon > 0$, we can find a positive integer h such that $\gamma_i^{p^h} - 1$ acts on $M_{[r/q,r]}$ with operator norm less than 1. On the other hand, for any μ which contributes a nonzero summand to $N_{[r/q,r],i,\psi}$ and any $(e,s) \in (\mathbb{Z} \times N^\vee)[\frac{1}{p}]$ lifting μ, for $x = \varphi^{-ad}((1+\pi)^e s)$, we have $(\gamma_i^{p^h} - 1)(x) = (1+\pi)^t x$ for some $t \in \mathbb{Z}[p^{-1}]$ of p-adic valuation at most $h + 1 - ad$. By taking a large enough, we can ensure that for $\mathbf{v} \in M_r$, the expression

$$(\gamma_i^{p^h} - 1)(x\mathbf{v}) = (\gamma_i^{p^h} - 1)(x)\mathbf{v} + \gamma_i^{p^h}(x)(\gamma_i^{p^h} - 1)(\mathbf{v})$$

from (4.3.2.1) is dominated by the term $(\gamma_i^{p^h} - 1)(x)\mathbf{v}$. This implies (by reducing to the boundary-free case as in Remark 4.4.2) that for a sufficiently large, $\gamma_i - 1$ acts bijectively on $M_r \otimes_{\mathbf{C}_\psi^r} \varphi^{-ad}(\mathbf{C}_{i,\psi}^{[r/q^{a+1}, r/q^a]}[\mu])$ with inverse bounded uniformly in μ. By Corollary 4.4.4, this implies the desired result. □

Lemma 4.7.10. *For d a positive integer, let M be a (φ^d, Γ_N)-module over $\mathbf{C}_\psi$. Then the continuous effective (φ^d, Γ_N)-cohomology groups of $M \otimes_{\mathbf{C}_\psi} (\tilde{\mathbf{C}}_\psi / \mathbf{C}_\psi)$ are all trivial.*

Proof. For the purposes of computing cohomology, [45, Proposition 6.4.5] allows to pass freely between $\tilde{\mathbf{C}}_\psi$ and $\tilde{\mathbf{C}}_\psi^{[r/q,r]}$, while Proposition 4.5.10 does the same between $\mathbf{C}_\psi$ and $\mathbf{C}_\psi^{[r/q,r]}$. Consequently, by Lemma 4.7.9, the continuous effective Γ_N-cohomology groups of $M \otimes_{\mathbf{C}_\psi} (\tilde{\mathbf{C}}_\psi / \varphi^{-ad}(\mathbf{C}_\psi))$ are trivial. On the other hand, it is clear that the φ^d-cohomology groups of $M \otimes_{\mathbf{C}_\psi} (\varphi^{-ad}(\mathbf{C}_\psi)/\mathbf{C}_\psi)$ are trivial. ☐

Finally, we note that the above results can also be established with $\mathbf{C}_\psi, \tilde{\mathbf{C}}_\psi$ replaced by $\mathbf{A}^\dagger_\psi, \tilde{\mathbf{A}}^\dagger_\psi$. The proofs are somewhat simpler in this case, so we can safely omit most details.

Proposition 4.7.11. *Let d be a positive integer and put $q = p^d$. Let M be a (φ^d, Γ_N)-module over $\tilde{\mathbf{A}}^\dagger_\psi$ whose underlying module is free. Then there exists a basis of M on which $\varphi, \gamma_0, \dots, \gamma_n$ act via matrices over $\mathbf{A}^\dagger_\psi$.*

Proof. Note that if we trace through the proof of Proposition 4.7.8 applied to $M \otimes_{\tilde{\mathbf{A}}^\dagger_\psi} \tilde{\mathbf{C}}_\psi$ using a basis of M, all of the coordinate changes produced by Lemma 4.7.6 are defined over $\tilde{\mathbf{A}}^\dagger_\psi$. ☐

Lemma 4.7.12. *For d a positive integer, let M be a (φ^d, Γ_N)-module over $\mathbf{A}^\dagger_\psi$. Then the continuous effective (φ^d, Γ_N)-cohomology groups of*

$$M \otimes_{\mathbf{A}^\dagger_\psi} (\tilde{\mathbf{A}}^\dagger_\psi / \mathbf{A}^\dagger_\psi), \qquad M \otimes_{\mathbf{A}^\dagger_\psi} (\tilde{\mathbf{B}}^\dagger_\psi / \mathbf{B}^\dagger_\psi), \qquad M \otimes_{\mathbf{A}^\dagger_\psi} (\tilde{\mathbf{C}}_\psi / \mathbf{C}_\psi)$$

are all trivial.

Proof. As in Lemma 4.7.10. ☐

4.8 Local systems and cohomolgoy

We are now ready to relate imperfect period rings to étale local systems and their cohomology.

Theorem 4.8.1. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, the categories of (φ^d, Γ_N)-modules over the rings in the diagram*

$$\begin{tikzcd}
\mathbf{A}^\dagger_\psi \arrow[r] \arrow[d] & \bigcup \varphi^{-n}(\mathbf{A}^\dagger_\psi) \arrow[r] \arrow[d] & \widehat{\bigcup} \varphi^{-n}(\mathbf{A}^\dagger_\psi) \arrow[r] \arrow[d] & \tilde{\mathbf{A}}^\dagger_\psi \arrow[d] \\
\mathbf{A}_\psi \arrow[r] & \bigcup \varphi^{-n}(\mathbf{A}_\psi) \arrow[r] & \widehat{\bigcup} \varphi^{-n}(\mathbf{A}_\psi) \arrow[r] & \tilde{\mathbf{A}}_\psi
\end{tikzcd}$$

are equivalent via the apparent base change functors, and are all equivalent to $\mathbb{Z}_{p^d}$-$\mathbf{Loc}(A)$ (and hence to $\mathbb{Z}_{p^d}$-$\mathbf{Loc}(\mathcal{M}(A))$ as in [45, Remark 2.8.2]).

Proof. The categories of (φ^d, Γ_N)-modules over $\tilde{\mathbf{A}}^\dagger_\psi$ and $\tilde{\mathbf{A}}_\psi$ are equivalent to each other and to $\mathbb{Z}_{p^d}$-$\mathbf{Loc}(A)$ by Theorem 3.6.1. It thus remains to produce enough equivalences of categories to link the rings in the right column to all of the other rings in the diagram. Among the rings in the bottom row of the diagram (which all carry the weak topologies), all of the claimed equivalences follow from Corollary 4.5.4.

We next check that the base change functors on (φ^d, Γ_N)-modules from all of the rings in the top row of the diagram down to $\tilde{\mathbf{A}}_\psi$ are fully faithful. It suffices to check the corresponding statement for φ^d-modules, which follows from Proposition 4.5.14(a) in the case of $\mathbf{A}^\dagger_\psi$ and from similar arguments in the other cases.

It finally remains to check that base extension of (φ^d, Γ_N)-modules from $\mathbf{A}^\dagger_\psi$ to $\mathbf{A}_\psi$ is essentially surjective. To check that a given (φ^d, Γ_N)-module M over $\mathbf{A}_\psi$ arises by base extension from a (φ^d, Γ_N)-module over $\mathbf{A}^\dagger_\psi$, it is enough to check after replacing A by a faithfully finite étale cover (by faithfully flat descent and Theorem 4.2.2). We may thus assume in addition that the corresponding $\mathbb{Z}_{p^d}$-local system T over A is constant modulo p.

Let $\tilde{M}$ be the (φ^d, Γ_N)-module over $\tilde{\mathbf{A}}^\dagger_\psi$ corresponding to $M \otimes_{\mathbf{A}_\psi} \tilde{\mathbf{A}}_\psi$. By our assumption about T, $\tilde{M}$ admits a basis fixed modulo p by φ^d and Γ_N. Let H be the subgroup of Γ_N indicated in Hypothesis 4.7.1. By Proposition 4.7.11, there exists a basis of $\tilde{M}$ on which φ^d and H act via matrices over $\mathbf{A}^\dagger_\psi$. In particular, the $\mathbf{A}^\dagger_\psi$-span M_0 of this basis is a φ^d-module over $\mathbf{A}^\dagger_\psi$. By Proposition 4.5.14(a), M_0 inherits an action of Γ_N; this action is effective and continuous because the same is true of the action on $M_0 \otimes_{\mathbf{A}^\dagger_\psi} \tilde{\mathbf{A}}^\dagger_\psi \cong \tilde{M}$. From the isomorphism $M_0 \otimes_{\mathbf{A}^\dagger_\psi} \tilde{\mathbf{A}}_\psi \cong M \otimes_{\mathbf{A}_\psi} \tilde{\mathbf{A}}_\psi$, we obtain an isomorphism $M_0 \otimes_{\mathbf{A}^\dagger_\psi} \mathbf{A}_\psi \cong M$ by applying Proposition 4.5.14(a) again. This yields the desired essential surjectivity. □

To consider $\mathbb{Q}_p$-local systems, we must as usual distinguish between the scheme $\operatorname{Spec}(A)$ and the analytic space $\mathcal{M}(A)$. We consider the former first.

Theorem 4.8.2. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, the categories of globally étale (φ^d, Γ_N)-modules over the rings in the diagram*

$$\begin{tikzcd}
\mathbf{C}_\psi \arrow[rrr] & & & \tilde{\mathbf{C}}_\psi \\
\mathbf{B}^\dagger_\psi \arrow[u] \arrow[d] \arrow[r] & \bigcup \varphi^{-n}(\mathbf{B}^\dagger_\psi) \arrow[d] \arrow[r] & \widehat{\bigcup} \varphi^{-n}(\mathbf{B}^\dagger_\psi) \arrow[d] \arrow[r] & \tilde{\mathbf{B}}^\dagger_\psi \arrow[u] \arrow[d] \\
\mathbf{B}_\psi \arrow[r] & \bigcup \varphi^{-n}(\mathbf{B}_\psi) \arrow[r] & \widehat{\bigcup} \varphi^{-n}(\mathbf{B}_\psi) \arrow[r] & \tilde{\mathbf{B}}_\psi
\end{tikzcd} \tag{4.8.2.1}$$

are equivalent via the apparent base change functors, and are all equivalent to $\mathbb{Q}_{p^d}$-$\mathbf{Loc}(A)$. *Consequently, for any $c \in \mathbb{Z}$, the categories of globally (c, d)-pure (φ, Γ_N)-modules over the rings in the diagram are equivalent via the apparent base change functors, and are all equivalent to the category of étale (c, d)-$\mathbb{Q}_p$-local systems on* $\operatorname{Spec}(A)$.

Proof. If we omit $\mathbf{C}_\psi$ and $\tilde{\mathbf{C}}_\psi$ from the diagram, then the claim follows from Theorem 4.8.1. We may join $\tilde{\mathbf{C}}_\psi$ to the rest of the diagram using Theorem 3.6.2. To connect $\mathbf{C}_\psi$ to the rest of the diagram, note that base extension of globally étale φ^d-modules from $\mathbf{B}^\dagger_\psi$ to $\mathbf{C}_\psi$ is an equivalence by Proposition 4.5.14(b). This gives full faithfulness of base extension of globally étale (φ^d, Γ_N)-modules from $\mathbf{B}^\dagger_\psi$ to $\mathbf{C}_\psi$. Also, any globally étale (φ^d, Γ_N)-module over $\mathbf{C}_\psi$ descends uniquely to a globally étale φ^d-module M over $\mathbf{B}^\dagger_\psi$ equipped with an action of Γ_N which commutes with φ^d, is effective, and is continuous for the LF topology. We may check continuity for the weak topology by extending scalars to $\tilde{\mathbf{C}}_\psi$ and invoking Theorem 3.6.2 again. This completes the proof. □

To handle the analytic space $\mathcal{M}(A)$, we must sheafify the previous construction; in so doing, we get to globalize to the case where ψ is not necessarily of strictly rational type but only strictly étale.

Definition 4.8.3. Throughout this definition, suppose that the frame ψ is strictly étale but not necessarily of rational type, and let d be a positive integer. Let ψ', ψ'' be frames of strictly rational type which factor through ψ. For φ-modules M_1, M_2 over the period rings $*_{\psi'}, *_{\psi''}$, we define a *local morphism* $M_1 \otimes_{*_{\psi'}} *_\psi \to M_2 \otimes_{*_{\psi''}} *_\psi$ of φ^d-modules to be a collection of morphisms $M_1 \otimes_{*_{\psi'}} *_{\psi_i} \to M_2 \otimes_{*_{\psi''}} *_{\psi_i}$ for some covering family $\psi_1, \dots, \psi_n$ of strictly rational subframes of ψ of rational type, subject to the condition of compatibility on overlaps. This last condition means that the morphisms defined on ψ_i and ψ_j agree on any rational subframe of both ψ_i and ψ_j of strictly rational type. (Such covering families exist thanks to Remark 2.2.6.) We consider a local morphism to be the same as the local morphism obtained by replacing the covering family with a refinement thereof. When ψ itself is of strictly rational type and $* = \mathbf{C}$, local morphisms are just morphisms in the usual sense by Corollary 4.5.8.

A *local φ^d-module* over $*_\psi$ is a glueing datum for φ^d-modules with respect to a covering family of rational subframes of strictly rational type (with the glueing maps given by local morphisms as in the previous paragraph). The natural functor from φ^d-modules to local φ^d-modules over any given period ring is fully faithful; it is an equivalence over $\mathbf{C}_\psi$ when ψ is of strictly rational type by Corollary 4.5.8 again. We define a *local (φ^d, Γ_N)-module* to be a local φ^d-module equipped with a compatible action of Γ_N which is effective and continuous for all available topologies, and pass along local properties of the underlying local φ^d-module such as being *étale* or *pure*.

Theorem 4.8.4. *Assume that ψ is strictly étale but not necessarily of strictly rational type. For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, the categories of étale local (φ^d, Γ_N)-modules over the rings in* (4.8.2.1) *are equivalent via the apparent base change functors, and are all equivalent to $\mathbb{Q}_{p^d}$-$\mathbf{Loc}(\mathcal{M}(A))$. Consequently, for any $c \in \mathbb{Z}$, the categories of (c,d)-pure (φ, Γ_N)-modules over the rings in the diagram are equivalent via the apparent base change functors, and are all equivalent to the category of étale (c,d)-$\mathbb{Q}_p$-local systems on $\mathcal{M}(A)$.*

Proof. This follows at once from Theorem 4.8.2. □

We next turn to étale cohomology.

Theorem 4.8.5. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, let T be an étale $\mathbb{Z}_{p^d}$-local system on* $\operatorname{Spec}(A)$. *Let M be the (φ^d, Γ_N)-module over one of $\tilde{\mathbf{A}}_\psi, \tilde{\mathbf{A}}_\psi^\dagger, \mathbf{A}_\psi, \mathbf{A}_\psi^\dagger$ corresponding to T via Theorem 4.8.1. Then for $i \geq 0$, there is a natural (in T and A) bijection $H^i_{\text{ét}}(\operatorname{Spec}(A), T) \cong H^i_{\varphi^d,\Gamma}(M)$.*

Proof. This follows from Theorem 3.6.4 using Lemma 4.7.12. □

Theorem 4.8.6. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, let E be an étale $\mathbb{Q}_{p^d}$-local system on* $\operatorname{Spec}(A)$. *Let M be the globally étale (φ^d, Γ_N)-module over one of $\tilde{\mathbf{B}}_\psi, \tilde{\mathbf{B}}_\psi^\dagger, \mathbf{B}_\psi, \mathbf{B}_\psi^\dagger, \mathbf{C}_\psi, \tilde{\mathbf{C}}_\psi$ corresponding to E via Theorem 4.8.2. Then for $i \geq 0$, there is a natural (in E and A) bijection $H^i_{\text{ét}}(\operatorname{Spec}(A), E) \cong H^i_{\varphi^d,\Gamma}(M)$.*

Proof. This follows from Theorem 3.6.5 plus Lemma 4.7.12. □

Theorem 4.8.7. *For d a positive integer such that $\mathbb{F}_{p^d} \subseteq K$, let E be an étale $\mathbb{Q}_{p^d}$-local system on $\mathcal{M}(A)$. Let M be the étale (φ^d, Γ_N)-module over one of $\mathbf{C}_\psi, \tilde{\mathbf{C}}_\psi$ corresponding to E via Theorem 3.6.3. Then for $i \geq 0$, there is a natural (in E and A) bijection $H^i_{\text{ét}}(\mathcal{M}(A), E) \cong H^i_{\varphi^d,\Gamma}(M)$.*

Proof. This follows from Theorem 3.6.6 plus Theorem 4.8.6. □

4.9 Some descent results

We have a descent result for (φ, Γ_N)-modules from perfect to imperfect period rings, which allows us to derive some facts about (φ, Γ_N)-modules over imperfect period rings by reduction to the perfect case.

Theorem 4.9.1. *Let d be a positive integer.*

(a) For M a (φ^d, Γ_N)-module over $\mathbf{C}_\psi$ and $i \geq 0$, the natural maps $H^i_{\varphi^d,\Gamma}(M) \to H^i_{\varphi^d,\Gamma}(M \otimes_{\mathbf{C}_\psi} \tilde{\mathbf{C}}_\psi)$ are bijections.

(b) Base extension of (φ^d, Γ_N)-modules from $\mathbf{C}_\psi$ to $\tilde{\mathbf{C}}_\psi$ is an equivalence of categories.

Proof. Part (a) follows by Lemma 4.7.10. For part (b), full faithfulness follows from the case $i = 0$ of (a). Thanks to full faithfulness and the glueing property for φ-modules over $\mathbf{C}_\psi$ (Corollary 4.5.8), we may deduce essential surjectivity locally, so we may fix $\gamma \in \mathcal{M}(A)$ and prove the claim after replacing ψ with a strictly rational subframe encircling γ. Using the fact that $\tilde{\mathcal{R}}_L^{[s,r]}$ is a principal ideal domain for any analytic field L [41, Proposition 2.6.8], we may reduce to the case of a (φ^d, Γ_N)-module over $\tilde{\mathbf{C}}_\psi$ represented by a finite free module over $\tilde{\mathbf{C}}_\psi^{[r/p^d,r]}$ for some $r > 0$ equipped with appropriate actions of φ and Γ_N. We may then invoke Proposition 4.7.8 to descend to a (φ^d, Γ_N)-module over $\mathbf{C}_\psi$. □

Corollary 4.9.2. *For d a positive integer, let M be a (φ^d, Γ_N)-module over $\mathbf{C}_\psi$. Then M is (globally) pure/étale if and only if $\tilde{M} = M \otimes_{*_\psi} \tilde{*}_\psi$ is (globally) pure/étale.*

Proof. Suppose first that $* = \mathbf{C}$. If $\tilde{M}$ is globally pure, then by Theorem 4.8.2 it descends uniquely to a globally pure (φ^d, Γ_N)-module N over $\mathbf{C}_\psi$. By Theorem 4.9.1(a), we obtain an isomorphism $M \cong N$, so M is globally pure. The other cases are similar. $\square$

Corollary 4.9.3. *For d a positive integer, let M be a (φ^d, Γ_N)-module over $\mathbf{C}_\psi$ such that the slope polygon of $\tilde{M} = M \otimes_{\mathbf{C}_\psi} \tilde{\mathbf{C}}_\psi$ is constant over $\mathcal{M}(\overline{A}_\psi)$. Then there exists a filtration $0 = M_0 \subset \cdots \subset M_l = M$ of M by (φ^d, Γ_N)-submodules such that M_i/M_{i-1} are vector bundles (and hence (φ^d, Γ_N)-modules) which are pure of constant slope, and $\mu(M_1/M_0) > \cdots > \mu(M_l/M_{l-1})$. In other words, the filtration of $\tilde{M}$ given in [45, Theorem 7.3.8] descends to M.*

Proof. Apply [45, Theorem 7.3.8] to $\tilde{M}$, then apply Theorem 4.9.1(b) and Corollary 4.9.2 to descend each step of the filtration. $\square$

We have the following analogue of [45, Remark 7.3.4].

Remark 4.9.4. Consider the following conditions on a (φ^d, Γ_N)-module M.

(a) The (φ^d, Γ_N)-module M is globally étale (i.e., admits a locally free étale model).

(b) The (φ^d, Γ_N)-module M admits an étale model.

(c) The (φ^d, Γ_N)-module M is étale (i.e., admits locally free local étale models).

(d) The (φ^d, Γ_N)-module M admits local etale models.

In all cases, (a) implies (b) and (c), which in turn each imply (d).

Over $\mathbf{B}_\psi$ or $\mathbf{B}^\dagger_\psi$, (d) implies (b). If A is normal, then (a) and (b) are equivalent as in [45, Proposition 8.1.13] (i.e., using Theorem 4.8.4), which implies that all four conditions are equivalent.

Over $\mathbf{C}_\psi$, (c) and (d) are equivalent by Corollary 4.9.2 plus [45, Theorem 7.3.6]. If A is normal, then again (a) and (b) are equivalent. However, the two pairs of conditions are not equivalent to each other, as in [45, Example 8.1.14].

4.10 Prior art

The results described here include a number of prior results, some of whose proofs may not immediately resemble the ones given here. We include some discussion of these results here.

Remark 4.10.1. In Theorem 4.8.4, if we take the frame $\psi : \mathcal{M}(K) \to \mathcal{M}(K)$ with σ the zero cone in the zero-dimensional vector space over $\mathbb{R}$, then the rings $\mathbf{B}_\psi, \mathbf{B}^\dagger_\psi, \mathbf{C}_\psi$ may be identified with $\mathcal{E}_K, \mathcal{R}^{\mathrm{bd}}_K, \mathcal{R}_K$.

Next, let L be a finite extension of K and consider the frame $\psi : \mathcal{M}(L) \to \mathcal{M}(K)$ with σ as in the previous case. The rings $\mathbf{B}^H_{\psi'}, (\mathbf{B}^\dagger_{\psi'})^H, \mathbf{C}^H_{\psi'}$ each have finitely many connected components, acted upon transitively by $\Gamma = \Gamma_N = \mathbb{Z}_p^\times$. Choose connected components $\mathbf{B}_L, \mathbf{B}^\dagger_L, \mathbf{C}_L$, respectively; each of them is stabilized by the subgroup Γ_L of Γ corresponding to $\mathrm{Gal}(L(\mu_{p^\infty})/L)$ via the cyclotomic character. The categories of étale (φ, Γ)-modules over $\mathbf{B}^H_{\psi'}, (\mathbf{B}^\dagger_{\psi'})^H, \mathbf{C}^H_{\psi'}$ may then be canonically identified with the categories of étale (φ, Γ_L)-modules over $\mathbf{B}_L, \mathbf{B}^\dagger_L, \mathbf{C}_L$. (The ring $\mathbf{C}_L$ is often denoted $\mathbf{B}^\dagger_{\mathrm{rig},L}$, e.g., in the work of Berger [8, 9].)

With these observations, the equivalence between $\mathbb{Z}_p$-$\mathbf{Loc}(A)$ and (φ, Γ_N)-modules over $\mathbf{A}_\psi$ in Theorem 4.8.1 becomes Fontaine's original theorem on (φ, Γ)-modules [27, Théorème 3.4.3]. (Fontaine uses a slightly different method to pass from $\mathbf{FÉt}(A_{\psi,\infty})$ to $\mathbf{FÉt}(A_\psi)$, based on the *field of norms* construction [28, 69]. This agrees with our construction thanks to [28, §3.4, Proposition].)

Remark 4.10.2. Generalizing the work of Fontaine, the equivalence between $\mathbb{Z}_p$-$\mathbf{Loc}(A)$ and (φ, Γ_N)-modules over $\mathbf{A}_\psi$ in Theorem 4.8.1 was established in some additional cases by Andreatta [4, Theorem 7.11]. A typical case to which Andreatta's work applies is $A = K\{T_1^\pm, \dots, T_n^\pm\}$. Similar results can be found in the work of Scholl [61].

The limitation of the applicability of Andreatta's method arises from a corresponding limitation in Faltings's *almost purity theorem* [24, 25]. Faltings's approach to the almost purity theorem is based on formal schemes, which makes it necessary to impose hypotheses in order to have reasonable choices of formal models of the affinoid spaces in question. However, the generalization of the field of norms correspondence introduced in [45, §3.6] (and independently by Scholze [62, 63]) eliminates any such reliance on integral models. We suspect that one can use the generalization of Faltings's theorem given by Gabber and Ramero [30, 31] to obtain results of the same strength, but we did not check this carefully.

Remark 4.10.3. Keep notation as in Remark 4.10.1. In Theorem 4.8.1, the equivalence between (φ, Γ_N)-modules over $\mathbf{A}^\dagger_\psi$ and $\mathbf{A}_\psi$ was originally established by Cherbonnier and Colmez [18, Corollaire III.5.2], using the Sen-Tate method of decompletion in continuous Galois cohomology. A more streamlined argument may be inferred from the generalization given by Berger and Colmez [11, Théorème 4.2.9]. Our method is similar in spirit but somewhat different in technical details, and leads to a simpler argument overall; see [44] for a presentation of the resulting proof of the Cherbonnier-Colmez theorem.

Remark 4.10.4. In those cases considered in Remark 4.10.2, the equivalence between (φ, Γ_N)-modules over $\mathbf{A}^\dagger_\psi$ and $\mathbf{A}_\psi$ in Theorem 4.8.1 was established by Andreatta and Brinon [5, Théorème 4.35]. As in Remark 4.10.3, the method is adapted from the Sen-Tate method, which we have short-circuited here.

Remark 4.10.5. Keep notation as in Remark 4.10.1. In Theorem 4.8.4, the equivalence between étale (φ, Γ_N)-modules over $\mathbf{B}^\dagger_\psi$ and $\mathbf{C}_\psi$ was originally observed by Berger [9, Proposition IV.2.2] but without discussion of the mismatch of topologies.

Remark 4.10.6. Keep notation as in Remark 4.10.1. In this case, the case of Theorem 4.8.5 over $\mathbf{A}_\psi$ was originally established by Herr [35]. The case over $\mathbf{A}^\dagger_\psi$, as well as Theorem 4.8.6, are due to the second author [53]. It would be interesting to use (φ, Γ_N)-modules to study cohomology of étale local systems on more general analytic spaces, e.g., to derive finiteness results.

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