Errata for *p-adic Differential Equations* (updated 24 Apr 2012)

Thanks to Atsushi Shiho for pointing out several errors, notably those in Lemma 10.3.6 and Theorem 13.6.1.

Theorem 1.4.9: the given proof is incorrect. A submultiplicative norm on a field need not be multiplicative; consider for instance the norm on $\mathbb{Q}$ given by taking the supremum of the p-adic norms. In fact, the given formula does not define a multiplicative norm in general; worse yet, it is possible to choose F so that there is no $\alpha \in E$ for which the given formula describes a multiplicative norm. (This is connected to the phenomenon of defect described in Chapter 3.)

Here is a correct proof of Theorem 1.4.9. For $\alpha \in E$, let $P(T)$ be the minimal polynomial of α over F, put $d = \deg(P)$, and define $|\alpha| = |P(0)|^{1/d}$. To check that this gives a multiplicative norm, choose an arbitrary $\beta \in E$ with minimal polynomial $Q(T)$ of degree f. The polynomials P and Q are irreducible, so by Theorem 2.2.1 their Newton polygons consist of single segments of some slopes r and s, respectively. Write $P(T) = \sum_i P_i T^i$ and $Q(T) = \sum_j Q_j T^j$; then $|P_i| \leq e^{-r(d-i)}$ and $|Q_j| \leq e^{-s(f-j)}$, with equality for $i = j = 0$.

Factor $P(T) = (T-\alpha_1)\cdots(T-\alpha_d)$ and $Q(T) = (T-\beta_1)\cdots(T-\beta_e)$ over some algebraic extension of F, and define

$$A(T) = \sum_k A_k T^k = \prod_{i=1}^{d}\prod_{j=1}^{e}(T-\alpha_i-\beta_j), \qquad M(T) = \sum_k M_k T^k = \prod_{i=1}^{d}\prod_{j=1}^{e}(T-\alpha_i\beta_j).$$

Then A_k is an integer polynomial in the P_i and Q_j which is homogeneous of degree $df-k$ for the weighting giving degree $d-i$ to P_i and degree $f-j$ to Q_j. This implies that $|A_k| \leq e^{-\min\{r,s\}(df-k)}$, so the Newton polygon of A has no slopes less than $\min\{r,s\}$. By the multiplicativity of Newton polygons, the same holds for the minimal polynomial of $\alpha+\beta$, so $|\alpha+\beta| \leq e^{-\min\{r,s\}} = \max\{|\alpha|,|\beta|\}$. Meanwhile, M_k is an integer polynomial in the P_i and Q_j which is homogeneous of bidegree $(df-k, df-k)$ for the weighting giving bidegree $(d-i,0)$ to P_i and bidegree $(0,f-j)$ to Q_j. This implies that $|M_k| \leq e^{-(r+s)(df-k)}$ with equality for $k=0$, so the Newton polygon of M has all slopes equal to $r+s$. By the multiplicativity of Newton polygons, the same holds for the minimal polynomial of $\alpha\beta$, so $|\alpha\beta| = e^{-r-s} = |\alpha||\beta|$.

Theorem 2.2.2: The hypothesis that R be a nonarchimedean ring includes the condition that the norm on R is multiplicative, which is too strong for some applications (e.g., to rings of matrices, as in Proposition 8.3.5). It should only be assumed that the norm on R is *submultiplicative*, i.e., for all $a,b \in R$, one has $|ab| \leq |a||b|$. Also, condition (d) should read $|ab-c| \leq \lambda^2|a||b|$, and the definition of B_μ should be correspondingly changed to

$$B_\mu = \{(u,v) \in U \times V : |(u,v)| \leq \mu|a||b|\}.$$

Definition 6.2.12: in the definition of a refined differential module, it is not necessary to assume *a priori* that V is pure, as this follows from the condition $|D|_{\mathrm{sp},V^\vee\otimes V} < |D|_{\mathrm{sp},V}$ by Lemma 6.2.8. Namely, if V is not pure, then it admits at least one subquotient V_1 with

$|D|_{\mathrm{sp},V_1} = |D|_{\mathrm{sp},V}$ and at least one subquotient V_2 with $|D|_{\mathrm{sp},V_2} < |D|_{\mathrm{sp},V}$. Then $V_2^\vee \otimes V_1$ occurs as a subquotient of $V^\vee \otimes V$, so $|D|_{\mathrm{sp},V^\vee \otimes V} \geq |D|_{\mathrm{sp},V_2^\vee \otimes V_1} = |D|_{\mathrm{sp},V}$.

Definition 10.3.1: The second displayed equation does not make sense, because pt^{p-1} is not an element of F'_ρ. It should instead read

$$D(v \otimes f) = D'(v) \otimes pt^{p-1} f + v \otimes d(f).$$

Lemma 10.3.6: The statements of (d) and (f) are not correct. To fix them, one must assume K contains the full group μ_p of p-th roots of unity. In this case, for $\zeta \in \mu_p$, define the map $\zeta : F_\rho \to F_\rho$ as the substitution $t \mapsto \zeta t$. For a finite differential module (V, D) over F_ρ, define the pullback $\zeta^*(V, D)$ as the differential module $(\zeta^* V, D')$ with

$$\zeta^* V = V \otimes_{F_\rho, \zeta} F_\rho, \qquad D'(v \otimes f) = D(v) \otimes \zeta f + v \otimes d(f).$$

Then the correct statement of (d) is that if $\mu_p \subseteq K$, then

$$\varphi^* \varphi_* V \cong \oplus_{\zeta \in \mu_p} \zeta^* V.$$

More precisely, the map $\zeta^* V \to \varphi^* \varphi_* V$ takes $v \otimes 1$ to $\sum_{i=0}^{p-1} (\zeta t)^i v \otimes t^{-1}$. Similarly, the correct statement of (f) is that if $\mu_p \subseteq K$, then

$$\varphi_* V_1 \otimes \varphi_* V_2 \cong \bigoplus_{\zeta \in \mu_p} \varphi_*(V_1 \otimes \zeta^* V_2).$$

Theorem 10.4.2: In the proof, in order to conclude that $V'' \otimes W_0$ is contained in a factor of $V' \otimes W_0$, it must be shown not only that $IR(V' \otimes W_m) = p^{-p/(p-1)}$, but also that $IR(X') = p^{-p/(p-1)}$ for every Jordan-Hölder constituent X' of $V' \otimes W_m$. Since $W_m \otimes W_{-m} \cong W_0$, we can write $X' = X \otimes W_m$ for $X = X' \otimes W_{-m}$. Then $IR(X) \geq IR(V) > p^{-p/(p-1)}$ by Lemma 9.4.6(a), so $IR(X') = p^{-p/(p-1)}$ by Lemma 9.4.6(c).

Theorem 10.5.1: The proof needs to be corrected to avoid the use of the incorrect formulation of Lemma 10.3.6(d) in the last paragraph (see above). This may be accomplished as follows.

Assume first that K contains the full group μ_p of p-th roots of unity. In the first sentence of the last paragraph of the proof, it is noted that $\varphi^* W'$ is a subquotient of $\varphi^* \varphi_* V$. By the corrected formulation of Lemma 10.3.6(d), the latter is isomorphic to $\oplus_{\zeta \in \mu_p} \zeta^* V$. Note that $IR(\zeta^* V) = IR(V)$ for each $\zeta \in \mu_p$ by Corollary 6.2.7. Since each $\zeta^* V$ is irreducible, each Jordan-Hölder constituent of $\varphi^* W'$ must be isomorphic to $\zeta^* V$ for some $\zeta \in \mu_p$, yielding $IR(\varphi^* W') = IR(V)$. We may then continue as in the original proof.

Still assuming that K contains μ_p, we may now deduce Proposition 10.6.1 and Theorem 10.6.2. To obtain Theorem 10.5.1 for general K, it is sufficient to verify that the subsidiary radii of V and $V \otimes_K K(\mu_p)$ coincide. For this, we may again assume V is irreducible. By Corollary 6.2.7, $IR(V) = IR(V \otimes_K K(\mu_p))$. This is not enough to deduce the desired result because $V \otimes_K K(\mu_p)$ may fail to be irreducible. However, by Theorem 10.6.2 applied over $K(\mu_p)$, $V \otimes_K K(\mu_p)$ admits a strong decomposition, which by Corollary 6.2.7

again is $\mathrm{Gal}(K(\mu_p)/K)$-invariant. The strong decomposition of $V \otimes_K K(\mu_p)$ must therefore contain only a single summand, from which the claim follows.

Proposition 10.6.1, Theorem 10.6.2: Note that for a given K, these results depend on Theorem 10.5.1 for that particular K. As noted above (see the correction to Theorem 10.5.1), we must first prove Theorem 10.5.1 assuming that K contains the full group of p-th roots of unity, then deduce Proposition 10.6.1 and Theorem 10.6.2 under this assumption, then deduce Theorem 10.5.1 in full, then deduce Proposition 10.6.1 and Theorem 10.6.2 in full.

Theorem 10.6.7: Since we may include μ_p in E, we may assume from the outset that $\mu_p \subset K$. In the proof that X is refined, the displayed equation should read

$$(\varphi_* X^\vee) \otimes (\varphi_* X) \cong \bigoplus_{\zeta \in \mu_p} \varphi_*(X^\vee \otimes \zeta^* X).$$

The rest of the proof continues unchanged.

Notes for Chapter 11: Footnote 1 is not quite accurate: Baldassarri's paper (F. Baldassarri, Continuity of the radius of convergence of differential equations on p-adic analytic curves, *Invent. Math.* **182** (2010), 513–584) only treats the radius of convergence, not all of the radii of optimal convergence.

Theorem 13.2.3: it should be assumed that the exponents of the regular singularity have p-adic non-Liouville differences, not that they are p-adic non-Liouville numbers.

Proposition 13.4.5: From what is written, it is unclear how Proposition 13.1.4 implies that $B_{\sigma_m(i)} - B_{\sigma_m(i+1)}$ is forced to be zero for m large, since it varies with m. The point is that we may apply Proposition 13.1.4 to each element of the finite set $\{B_j - B_k : 1 \leq j, k \leq n\}$ to obtain a uniform lower bound $m_0 > 0$ such that for $1 \leq j, k \leq n$ and $m \geq m_0$, if $B_j \neq B_k$, then $|B_j - B_k|^{(m)} > 3cm \geq 2cm + c$. We then apply this with $j = \sigma_m(i), k = \sigma_{m+1}(i)$.

Definition 13.5.2: delete "w" at the end of page 227, line 10.

Theorem 13.5.6: The definition of T_m should read $T_m = S_{m,A}^{-1} S_{m,B}$. The quantity $T_{i,\sigma_m(i)}$ is never defined: it is shorthand for the matrix entry $(T_m)_{i,\sigma_m(i)}$. In the last displayed equation, $A_1, \dots, A_n, B_1, \dots, B_n$ should be $A_1^{(m)}, \dots, A_n^{(m)}, B_1^{(m)}, \dots, B_n^{(m)}$, respectively.

Theorem 13.6.1: The given proof is incorrect starting from the third paragraph: the upper bound on $|T_{m',m}|_\rho, |T_{m',m}^{-1}|_\rho$ should be $p^{nkm'}$ rather than p^{nkm}. The argument from this point should instead be carried out as follows.

As in the proof of Theorem 13.5.6, for all m we have

$$|S_{m,A}|_\rho \leq p^{km}, \quad |S_{m,A}^{-1}|_\rho \leq p^{(n-1)km} \qquad (\rho \in [\alpha'', \beta'']).$$

Choose $\lambda \in (0,1)$ and $c > 0$ so that $p^{8nk}\eta^{-c} \leq \lambda$. Since A has p-adic non-Liouville differences, there exists $m_0 > 0$ such that for $m \geq m_0$, the congruence $h \equiv A_i - A_j \pmod{p^m}$ forces either $h = A_i - A_j = 0$ or $|h| \geq cm$. By enlarging m_0 if needed, we may also ensure that $A_i \equiv A_j \pmod{p^{m_0}}$ if and only if $A_i = A_j$.

The strategy is to renormalize the matrices $S_{m,A}$ to obtain a convergent sequence. To this end, we will construct invertible matrices R_m over K for $m \geq m_0$ such that $R_{m_0} = I_n$, $(R_m)_{ij} = 0$ whenever $A_i \neq A_j$ (or equivalently $A_i \not\equiv A_j \pmod{p^m}$), and

$$|I_n - R_m S_{m,A}^{-1} S_{m+1,A} R_{m+1}^{-1}|_\rho \leq \lambda^m \qquad (\rho \in [\alpha', \beta'], m \geq m_0).$$

This will imply that for $m \geq m_0$ and $\rho \in [\alpha', \beta']$, $|I_n - S_{m_0,A}^{-1} S_{m,A} R_m^{-1}|_\rho < 1$ and $|S_{m_0,A}^{-1} S_{m,A} R_m^{-1} - S_{m_0,A}^{-1} S_{m+1,A} R_{m+1}^{-1}|_\rho \leq \lambda^m$. Consequently, the sequence $S_{m_0,A}^{-1} S_{m,A} R_m^{-1}$ for $m = m_0, m_0+1, \dots$ will converge to an invertible matrix U over $K\langle \alpha'/t, t/\beta' \rangle$ such that $S_{m_0,A} U$ is the change-of-basis matrix to a basis of $M \otimes K\langle \alpha'/t, t/\beta' \rangle$ of the desired form. This will complete the proof.

The construction of the R_m proceeds recursively as follows. Given $R_{m_0}, \dots, R_m$, we first verify that

$$|R_m|, |R_m^{-1}| \leq p^{nkm}.$$

This is clear for $m = m_0$, so we may assume $m > m_0$. Choose any $\rho \in [\alpha', \beta']$. As noted above, we have $|I_n - S_{m_0,A}^{-1} S_{m,A} R_m^{-1}|_\rho < 1$, so $|S_{m_0,A}^{-1} S_{m,A} R_m^{-1}|_\rho = |R_m S_{m,A}^{-1} S_{m_0,A}|_\rho = 1$. We then deduce the claim using the bounds on $|S_{m,A}|_\rho, |S_{m,A}^{-1}|_\rho$ from above.

Next, put $T_m = R_m S_{m,A}^{-1} S_{m+1,A}$; we then have $|T_m|_\rho, |T_m^{-1}|_\rho \leq p^{2nk(m+1)}$ for all $\rho \in [\alpha'', \beta'']$. If we write $T_m = \sum_{h \in \mathbb{Z}} T_{m,h} t^h$, then $(T_{m,h})_{ij}$ can only be nonzero if $h \equiv A_i - A_j \pmod{p^m}$, which forces either $h = 0$ or $|h| \geq cm$. If $h > 0$, we have

$$|(T_{m,h})_{ij} t^h|_{\alpha'} \leq |(T_{m,h})_{ij} t^h|_{\beta'} \leq |(T_{m,h})_{ij} t^h|_{\beta''} \eta^{-cm} \leq p^{2nk(m+1)} \eta^{-cm},$$

while if $h < 0$, we have

$$|(T_{m,h})_{ij} t^h|_{\beta'} \leq |(T_{m,h})_{ij} t^h|_{\alpha'} \leq |(T_{m,h})_{ij} t^h|_{\alpha''} \eta^{-cm} \leq p^{2nk(m+1)} \eta^{-cm}.$$

We may now take $R_{m+1} = T_{m,0}$, because

$$|I_n - R_{m+1} T_m^{-1}|_\rho \leq |T_m^{-1}|_\rho |T_m - T_{m,0}|_\rho \leq p^{2nk(m+1)} p^{2nk(m+1)} \eta^{-cm} \leq \lambda^m < 1 \qquad (\rho \in [\alpha', \beta'])$$

and so $|I_n - T_m R_{m+1}^{-1}|_\rho \leq \lambda^m$. (Note that indeed $(R_m)_{ij} = 0$ whenever $A_i \not\equiv A_j \pmod{p^m}$, and that the latter condition is equivalent to $A_i \neq A_j$ by our choice of m_0.) This completes the construction of the R_m and thus the proof.

Exercises for Chapter 13: in exercise 10, it should be assumed that M has p-adic non-Liouville exponent differences.

Section 18.2: at the end of the first sentence, the ring should be $K\{t/\beta\}$.

Corollary 19.4.2: $G_{\kappa_K((t)),i}$ should be $G^i_{\kappa_K((t))}$.