

NEW METHODS FOR (φ, Γ) -MODULES

KIRAN S. KEDLAYA

Let p be a prime number. The subject of *p -adic Hodge theory* concerns the interplay between different objects arising from the cohomology of algebraic varieties over p -adic fields. A good introduction to the subject can be found in the notes of Brinon and Conrad [3]; here, we limit discussion to two proofs which we feel are simpler than their counterparts in the literature. Our original motivation for obtaining such proofs was to facilitate generalization of the existing results; such generalizations to the context of relative p -adic Hodge theory can be found in our joint work with Liu [11, 12], and in closely related work of Scholze [13, 14].

Our first topic is the relationship between Galois theory in characteristic 0 and characteristic p provided by a theorem of Fontaine and Wintenberger [7].

Theorem 0.0.1 (Fontaine-Wintenberger). *For μ_{p^∞} the group of all p -power roots of unity in an algebraic closure of \mathbb{Q}_p , the absolute Galois groups of the fields $\mathbb{F}_p((\bar{\pi}))$ and $\mathbb{Q}_p(\mu_{p^\infty})$ are isomorphic (and even homeomorphic as profinite topological groups).*

The original proof of Theorem 0.0.1 depends in a crucial way on higher ramification theory of local fields, as developed for instance in the book of Serre [16]. This causes difficulties when trying to generalize Theorem 0.0.1, e.g., to local fields with imperfect residue fields. We expose here a new approach to Theorem 0.0.1 in which ramification theory plays no role; one instead makes a careful analysis of rings of Witt vectors over valuation rings. This leads naturally to a more general result, in which $\mathbb{Q}_p(\mu_{p^\infty})$ can be replaced by any sufficiently ramified p -adic field; this generalization is central to Scholze's study of *perfectoid spaces* [13, 14], which has strong echoes in [11, 12]. (Scholze gives a slightly different proof of the same generalization of Theorem 0.0.1; see Remark 1.5.7.)

Our second topic is the description of continuous representations of p -adic Galois groups on \mathbb{Q}_p -vector spaces (such as might arise from étale cohomology with p -adic coefficients) in terms of (φ, Γ) -modules. The original description of this form was given by Fontaine [6] in terms of a Cohen ring for a field of formal power series, and is an easy consequence of Theorem 0.0.1. Our main focus is the refinement of Fontaine's result by Cherbonnier and Colmez [4], in which the Cohen ring is replaced with a somewhat smaller ring of convergent power series (see Theorem 2.6.2 for the precise statement). This refinement is critical to a number of applications of p -adic Hodge theory (which we do not discuss here, but see [1]).

Existing proofs of the Cherbonnier-Colmez theorem, including a generalization to families of representations by Berger and Colmez [2], rely on some calculations involving a formalism for decompletion in continuous Galois cohomology, inspired by results of Tate and Sen and later axiomatized by Colmez. However, one can express the proof in such a way that one makes essentially the same calculations on (φ, Γ) -modules as in [4], but without any need to introduce the Tate-Sen formalism. We hope that this makes the proof somewhat more

Date: in progress; version of January 23, 2012.

Supported by NSF CAREER grant DMS-0545904, DARPA grant HR0011-09-1-0048, MIT (NEC Fund), UC San Diego (Warschawski Professorship). Thanks to Ryan Rodriguez for helpful feedback.

transparent. (It would be an interesting exercise to reprove the main result of [2] in this fashion.)

1. COMPARISON OF GALOIS GROUPS

1.1. Preliminaries on strict p -rings. We begin by recalling some basic properties of strict p -rings underlying the constructions made later, following the derivations in [16, §5]. (All rings considered will be commutative and unital.)

Lemma 1.1.1. *For any ring R and any nonnegative integer n , the map $x \mapsto x^{p^n}$ induces a well-defined multiplicative monoid map $\theta_n : R/(p) \rightarrow R/(p^{n+1})$.*

Proof. If $x \equiv y \pmod{p^m}$ for some positive integer m , then $x^p - y^p = (x - y)(x^{p-1} + \dots + y^{p-1})$ and the latter factor is congruent to px^{p-1} modulo p^m ; hence $x^p \equiv y^p \pmod{p^{m+1}}$. This proves that θ_n is well-defined; it is clear that it is multiplicative. \square

Definition 1.1.2. A ring R of characteristic p is *perfect* if the Frobenius homomorphism $x \mapsto x^p$ is a bijection; this forces R to be reduced. (If R is a field, then R is perfect if and only if every finite extension of R is separable.) A *strict p -ring* is a p -torsion-free, p -adically complete ring R for which $R/(p)$ is perfect, regarded as a topological ring using the p -adic topology.

Example 1.1.3. The ring \mathbb{Z}_p is a strict p -ring with $\mathbb{Z}_p/(p) \cong \mathbb{F}_p$. Similarly, for any (possibly infinite) set X , if we write $\mathbb{Z}[X]$ for the polynomial ring over \mathbb{Z} generated by X and put $\mathbb{Z}[X^{p^{-\infty}}] = \cup_{n=0}^{\infty} \mathbb{Z}[X^{p^{-n}}]$, then the p -adic completion R of $\mathbb{Z}[X^{p^{-\infty}}]$ is a strict p -ring with $R/(p) \cong \mathbb{F}_p[X^{p^{-\infty}}]$.

Lemma 1.1.4. *Let \bar{R} be a perfect ring of characteristic p , let S be a p -adically complete ring, and let $\pi : S \rightarrow S/(p)$ be the natural projection. Let $\bar{t} : \bar{R} \rightarrow S/(p)$ be a ring homomorphism. Then there exists a unique multiplicative map $t : \bar{R} \rightarrow S$ with $\pi \circ t = \bar{t}$. In fact, $t(\bar{x}) \equiv x_n^{p^n} \pmod{p^{n+1}}$ for any nonnegative integer n and any $x_n \in S$ lifting $\bar{t}(\bar{x}^{p^{-n}})$.*

Proof. This is immediate from Lemma 1.1.1. \square

Definition 1.1.5. By the case $S = R$ of Lemma 1.1.4, the projection $R \rightarrow R/(p)$ admits a unique multiplicative section $[\cdot] : R/(p) \rightarrow R$, called the *Teichmüller map*. (For example, the image of $[\cdot] : \mathbb{F}_p \rightarrow \mathbb{Z}_p$ consists of 0 together with the $(p - 1)$ -st roots of unity in \mathbb{Z}_p .) Each $x \in R$ admits a unique representation as a p -adically convergent sum $\sum_{n=0}^{\infty} p^n [\bar{x}_n]$ for some elements $\bar{x}_n \in R/(p)$, called the *Teichmüller coordinates* of x .

Lemma 1.1.6. *Let R be a strict p -ring, let S be a p -adically complete ring, and let $\pi : S \rightarrow S/(p)$ be the natural projection. Let $t : R/(p) \rightarrow S$ be a multiplicative map such that $\bar{t} = \pi \circ t$ is a ring homomorphism. Then the formula*

$$(1.1.6.1) \quad T \left(\sum_{n=0}^{\infty} p^n [\bar{x}_n] \right) = \sum_{n=0}^{\infty} p^n t(\bar{x}_n) \quad (\bar{x}_0, \bar{x}_1, \dots \in R/(p))$$

defines a (necessarily unique) homomorphism $T : R \rightarrow S$ such that $T \circ [\cdot] = t$.

Proof. We check by induction that for each positive integer n , T induces an additive map $R/(p^n) \rightarrow S/(p^n)$. This holds for $n = 1$ because $\pi \circ t$ is a homomorphism. Suppose the claim holds for some $n \geq 1$. For $x = [\bar{x}] + px_1, y = [\bar{y}] + py_1, z = [\bar{z}] + pz_1 \in R$ with $x + y = z$,

$$\begin{aligned} [\bar{z}] &\equiv ([\bar{x}^{p^{-n}}] + [\bar{y}^{p^{-n}}])^{p^n} \pmod{p^{n+1}} \\ t(\bar{z}) &\equiv (t(\bar{x}^{p^{-n}}) + t(\bar{y}^{p^{-n}}))^{p^n} \pmod{p^{n+1}} \end{aligned}$$

by Lemma 1.1.4. In particular,

$$(1.1.6.2) \quad T([\bar{z}]) - T([\bar{x}]) - T([\bar{y}]) \equiv \sum_{i=1}^{p^n-1} \binom{p^n}{i} t(\bar{x}^{ip^{-n}} \bar{y}^{1-ip^{-n}}) \pmod{p^{n+1}}.$$

On the other hand, since $\frac{1}{p} \binom{p^n}{i} \in \mathbb{Z}$ for $i = 1, \dots, p^n - 1$, we may write

$$z_1 - x_1 - y_1 = \frac{[\bar{x}] + [\bar{y}] - [\bar{z}]}{p} \equiv - \sum_{i=1}^{p^n-1} \frac{1}{p} \binom{p^n}{i} [\bar{x}^{ip^{-n}} \bar{y}^{1-ip^{-n}}] \pmod{p^n},$$

apply T , invoke the induction hypothesis on both sides, and multiply by p to obtain

$$(1.1.6.3) \quad pT(z_1) - pT(x_1) - pT(y_1) \equiv - \sum_{i=1}^{p^n-1} \binom{p^n}{i} t(\bar{x}^{ip^{-n}} \bar{y}^{1-ip^{-n}}) \pmod{p^{n+1}}.$$

Since $T(x) = T([\bar{x}]) + pT(x_1)$ and so on, we may add (1.1.6.2) and (1.1.6.3) to deduce that $T(z) - T(x) - T(y) \equiv 0 \pmod{p^{n+1}}$, completing the induction. Hence T is additive; multiplicativity of t forces T to also be multiplicative, as desired. \square

Remark 1.1.7. Take R as in Example 1.1.3 with $X = \{\bar{x}, \bar{y}\}$. By Lemma 1.1.4, we have

$$(1.1.7.1) \quad [\bar{x}] - [\bar{y}] = \sum_{n=0}^{\infty} p^n [P_n(\bar{x}, \bar{y})]$$

for some $P_n(\bar{x}, \bar{y})$ in the ideal $(\bar{x}^{p^{-\infty}}, \bar{y}^{p^{-\infty}}) \subset \mathbb{F}_p[\bar{x}^{p^{-\infty}}, \bar{y}^{p^{-\infty}}]$ and homogeneous of degree 1. By Lemma 1.1.6, (1.1.7.1) is also valid for any strict p -ring R and any $\bar{x}, \bar{y} \in R/(p)$. One can similarly derive formulas for arithmetic in a strict p -ring in terms of Teichmüller coordinates; these can also be obtained using Witt vectors (Definition 1.1.9).

Theorem 1.1.8. *The functor $R \rightsquigarrow R/(p)$ from strict p -rings to perfect rings of characteristic p is an equivalence of categories.*

Proof. Full faithfulness follows from Lemma 1.1.6. To prove essential surjectivity, let \bar{R} be a perfect ring of characteristic p , choose a surjection $\psi : \mathbb{F}_p[X^{p^{-\infty}}] \rightarrow \bar{R}$ for some set X , and put $\bar{I} = \ker(\psi)$. Let R_0 be the p -adic completion of $\mathbb{Z}[X^{p^{-\infty}}]$; as in Example 1.1.3, this is a strict p -ring with $R_0/(p) \cong \mathbb{F}_p[X^{p^{-\infty}}]$. Put $I = \{\sum_{n=0}^{\infty} p^n [\bar{x}_n] \in R_0 : \bar{x}_0, \bar{x}_1, \dots \in \bar{I}\}$; this forms an ideal in R_0 by Remark 1.1.7. Then $R = R_0/I$ is a strict p -ring with $R/(p) \cong \bar{R}$. \square

Definition 1.1.9. For \bar{R} a perfect ring of characteristic p , we write $W(\bar{R})$ for the unique (by Theorem 1.1.8) strict p -ring with $W(\bar{R})/(p) \cong \bar{R}$. This is meant as a reminder that $W(\bar{R})$ also occurs as the ring of p -typical Witt vectors over \bar{R} ; that construction obtains the formulas for arithmetic in Teichmüller coordinates in an elegant manner linked to symmetric functions.

1.2. Perfect norm fields. Theorem 0.0.1 is obtained by matching up the Galois correspondences of the fields $\mathbb{F}_p((\bar{\pi}))$ and $\mathbb{Q}_p(\mu_{p^\infty})$. The approach taken by Fontaine and Wintenberger is to pass from characteristic 0 to characteristic p by looking at certain sequences of elements of finite extensions of \mathbb{Q}_p in which each term is obtained from the succeeding term by taking a certain norm between fields; the resulting functor is thus commonly called the functor of *norm fields*. It is here that some careful analysis of higher ramification theory is needed in order to make the construction work.

While the Fontaine-Wintenberger construction gives rise directly to finite extensions of $\mathbb{F}_p((\bar{\pi}))$, it was later observed that a simpler construction (used repeatedly by Fontaine in his further study of p -adic Hodge theory) could be used to obtain the perfect closures of these finite extensions. Originally the construction of these *perfect norm fields* depended crucially on the prior construction of the *imperfect norm fields* of Fontaine-Wintenberger (as in the exposition in [3]), but we will instead work directly with the perfect norm fields.

Definition 1.2.1. By an *analytic field*, we will mean a field K which is complete with respect to a multiplicative nonarchimedean norm $|\cdot|$. For K an analytic field, put $\mathfrak{o}_K = \{x \in K : |x| \leq 1\}$; this is a local ring with maximal ideal $\mathfrak{m}_K = \{x \in K : |x| < 1\}$. We say K has *mixed characteristics* if $|p| = p^{-1}$ (so K has characteristic 0) and the residue field $\kappa_K = \mathfrak{o}_K/\mathfrak{m}_K$ of K has characteristic p .

Remark 1.2.2. Any finite extension L of an analytic field K is itself an analytic field; that is, the norm extends uniquely to a multiplicative norm on the extension field. A key fact about this extension is *Krasner's lemma*: if $P(T) \in K[T]$ splits over L as $\prod_{i=1}^n (T - \alpha_i)$ and $\beta \in K$ satisfies $|\alpha_1 - \beta| < |\alpha_1 - \alpha_i|$ for $i = 2, \dots, n$, then $\alpha_1 \in K$.

A crucial consequence of Krasner's lemma is that an infinite algebraic extension of K is separably closed if and only if its completion is algebraically closed. More precisely, Krasner's lemma is only needed for the "if" implication; the "only if" implication follows from the fact that the roots of a polynomial vary continuously in the coefficients. This principle also appears in the proof of Lemma 1.5.4.

Remark 1.2.3. Using Krasner's lemma, one may show that $\mathbb{Q}_p(\mu_{p^\infty})$ and its completion have the same Galois group. Similarly, $\mathbb{F}_p((\bar{\pi}))$, its perfect closure, and the completion of the perfect closure all have the same Galois group. Consequently, from the point of view of proving Theorem 0.0.1, there is no harm in considering only analytic fields.

Definition 1.2.4. Let K be an analytic field of mixed characteristics. Let $\mathfrak{o}_{K'}$ be the inverse limit $\varprojlim \mathfrak{o}_K/(p)$ under Frobenius. In symbols,

$$\mathfrak{o}_{K'} = \left\{ (\bar{x}_n) \in \prod_{n=0}^{\infty} \mathfrak{o}_K/(p) : \bar{x}_{n+1}^p = \bar{x}_n \right\}.$$

By construction, this is a perfect ring of characteristic p : the inverse of Frobenius is the shift map $(\bar{x}_n)_{n=0}^{\infty} \mapsto (\bar{x}_{n+1})_{n=0}^{\infty}$. By applying Lemmas 1.1.4 and 1.1.6 to the homomorphism $\bar{\theta} : \mathfrak{o}_{K'} \rightarrow \mathfrak{o}_K/(p)$, we obtain a multiplicative map $\theta : \mathfrak{o}_{K'} \rightarrow \mathfrak{o}_K$ and a homomorphism $\Theta : W(\mathfrak{o}_{K'}) \rightarrow \mathfrak{o}_K$.

For $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots) \in \mathfrak{o}_{K'}$, define $|\bar{x}'| = |\theta(\bar{x})|$. If we lift $\bar{x}_n \in \mathfrak{o}_K/(p)$ to $x_n \in \mathfrak{o}_K$, then $|\bar{x}'| = |x_n|^{p^n}$ whenever $|x_n| > |p|$. Consequently, $|\cdot|'$ is a multiplicative nonarchimedean norm on $\mathfrak{o}_{K'}$ under which $\mathfrak{o}_{K'}$ is complete (given any Cauchy sequence, the terms in any particular position eventually stabilize).

Exercise 1.2.5. The formula $x \mapsto (\theta(x^{p^{-n}}))_{n=0}^\infty$ defines a multiplicative bijection from $\mathfrak{o}_{K'}$ to the inverse limit of multiplicative monoids (but not of rings)

$$\left\{ (x_n) \in \prod_{n=0}^\infty \mathfrak{o}_K : x_{n+1}^p = x_n \right\}.$$

Lemma 1.2.6. *With notation as in Definition 1.2.4, for $\bar{x}, \bar{y} \in \mathfrak{o}_{K'}$, \bar{x} is divisible by \bar{y} if and only if $|\bar{x}'| \leq |\bar{y}'|$.*

Proof. If \bar{x} is divisible by \bar{y} , then $|\bar{x}'| \leq |\bar{y}'| |\bar{x}/\bar{y}'| \leq |\bar{y}'|$. Conversely, suppose $|\bar{x}'| \leq |\bar{y}'|$. If $\bar{y} = 0$, then $\bar{x} = 0$ also and there is nothing more to check. Otherwise, write $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots)$, $\bar{y} = (\bar{y}_0, \bar{y}_1, \dots)$, and choose lifts x_n, y_n of \bar{x}_n, \bar{y}_n to \mathfrak{o}_K . Since $\bar{y} \neq 0$, we can find an integer $n_0 \geq 0$ such that $|y_n| \geq p^{-1+1/p}$ for $n = n_0$, and hence also for $n \geq n_0$. Then for $n \geq n_0$, the elements $z_n = x_n/y_n \in \mathfrak{o}_K$ have the property that $|z_{n+1}^p - z_n| \leq p^{-1/p}$. By writing $z_{n+2}^{p^2} = (z_{n+1} + (z_{n+2}^p - z_{n+1}^p))^p$, we deduce that $|z_{n+2}^{p^2} - z_{n+1}^p| \leq p^{-1}$. We thus produce an element $\bar{z} = (\bar{z}_0, \bar{z}_1, \dots)$ with $\bar{x} = \bar{y}\bar{z}$ by taking \bar{z}_n to be the reduction of z_{n+1}^p for $n \geq n_0 + 1$. \square

Definition 1.2.7. Keep notation as in Definition 1.2.4. By Lemma 1.2.6, $\mathfrak{o}_{K'}$ is the valuation ring in an analytic field K' which is perfect of characteristic p . We call K' the *perfect norm field* associated to K . (Scholze [13] calls K' the *tilt* of K .)

1.3. Perfectoid fields. In general, the perfect norm field functor is far from being faithful.

Exercise 1.3.1. Let K be a discretely valued analytic field of mixed characteristics. Then K' is isomorphic to the maximal perfect subfield of κ_K .

To get around this problem, we restrict attention to analytic fields with a great deal of ramification. (The term *perfectoid* is due to Scholze [13].)

Definition 1.3.2. An analytic field K is *perfectoid* if K is of mixed characteristics, K is not discretely valued, and the p -th power Frobenius endomorphism on $\mathfrak{o}_K/(p)$ is surjective.

The following statements imply that taking the perfect norm field of a perfectoid analytic field does not concede too much information.

Lemma 1.3.3. *Let K be a perfectoid analytic field.*

- (a) *We have $|K^\times| = |(K')^\times|$.*
- (b) *The projection $\mathfrak{o}_{K'} \rightarrow \varprojlim \mathfrak{o}_K/(p)$ is surjective and induces an isomorphism $\mathfrak{o}_{K'}/(\bar{z}) \cong \mathfrak{o}_K/(p)$ for any $\bar{z} \in \mathfrak{o}_{K'}$ with $|\bar{z}'| = p^{-1}$ (which exists by (a)). In particular, we obtain a natural isomorphism $\kappa_{K'} \cong \kappa_K$.*
- (c) *The map $\Theta : W(\mathfrak{o}_{K'}) \rightarrow \mathfrak{o}_K$ is also surjective. (The kernel of Θ turns out to be a principal ideal; see Corollary 1.4.14.)*

Proof. Since K is not discretely valued, we can find $r \in |K^\times|$ such that $p^{-1}r^p \in (p^{-1}, 1)$; the surjectivity of Frobenius then implies $p^{-1/p}r \in |K^\times|$. This implies $p^{-1/p} \in |K^\times|$, so $p^{-1/p} \in |(K')^\times|$ and $p^{-1} \in |(K')^\times|$. Since also $|K^\times| \cap (p^{-1}, 1) = |(K')^\times| \cap (p^{-1}, 1)$, we obtain (a), and (b) and (c) follow easily. \square

Corollary 1.3.4. *The perfect norm field functor on perfectoid analytic fields is faithful.*

Example 1.3.5. Take K to be the completion of $\mathbb{Q}_p(\mu_{p^\infty})$ for the unique extension of the p -adic norm, and fix a choice of a sequence $\{\zeta_{p^n}\}_{n=0}^\infty$ in which ζ_{p^n} is a primitive p^n -th root of unity and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. The field K is perfectoid because

$$\mathfrak{o}_K/(p) \cong \mathbb{F}_p[\bar{\zeta}_p, \bar{\zeta}_{p^2}, \dots]/(1 - \bar{\zeta}_p^p, \bar{\zeta}_p - \bar{\zeta}_{p^2}^p, \dots).$$

By the same calculation, we identify K' with the completed perfect closure of $\mathbb{F}_p((\bar{\pi}))$ by identifying $\bar{\pi}$ with $(\bar{\zeta}_1 - 1, \bar{\zeta}_p - 1, \dots)$. This example underlies the theory of (φ, Γ) -modules; see §2.

Example 1.3.6. Let F be a discretely valued analytic field of mixed characteristics (e.g., a finite extension of \mathbb{Q}_p) and choose a uniformizer π of F . Choose a sequence π_0, π_1, \dots of elements of an algebraic closure of F in which $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$ for $n \geq 0$. Take K to be the completion of $F(\pi_0, \pi_1, \dots)$; then \mathfrak{o}_K is the completion of $\mathfrak{o}_F[\pi_1, \pi_2, \dots]$, so

$$\mathfrak{o}_K/(p) \cong \mathbb{F}_p[\bar{\pi}_1, \bar{\pi}_2, \dots]/(\bar{\pi}_1^p, \bar{\pi}_1 - \bar{\pi}_2^p, \dots).$$

Consequently, K is perfectoid. By the same calculation, we identify K' with the completed perfect closure of $\mathbb{F}_p((\bar{\pi}))$ by identifying $\bar{\pi}$ with $(\bar{\pi}_0, \bar{\pi}_1, \dots)$. This example underlies the theory of *Breuil-Kisin modules*, which are a good replacement for (φ, Γ) -modules for the study of crystalline representations; see [3, §11].

Exercise 1.3.7. Let K be an analytic field of mixed characteristics which is not discretely valued.

- (a) Assume that there exists $\xi \in K$ with $p^{-1} \leq |\xi| < 1$ such that Frobenius is surjective on $\mathfrak{o}_K/(\xi)$. Then K is perfectoid. (Hint: imitate the proof of Lemma 1.2.6.)
- (b) Suppose that there exists an ideal $I \subseteq \mathfrak{m}_K$ such that the I -adic topology and the norm topology on \mathfrak{o}_K coincide, and Frobenius is surjective on \mathfrak{o}_K/I . Then K is perfectoid. (Note that I need not be principal; for instance, take $I = \{x \in \mathfrak{o}_K : |x| < p^{-1/2}\}$. If I is finitely generated, however, it is principal.)

Exercise 1.3.8. An analytic field K of mixed characteristics is perfectoid if and only if for every $x \in K$, there exists $y \in K$ with $|y^p - x| \leq p^{-1}|x|$. As in the previous exercise, one can also replace p^{-1} by any constant value in the range $[p^{-1}, 1)$.

Remark 1.3.9. When developing the theory of norm fields, it is typical to consider the class of *arithmetically profinite* algebraic extensions of \mathbb{Q}_p , i.e., those for which the Galois closure has the property that its higher ramification subgroups are open. One then shows using Exercise 1.3.7 that the completions of such extensions are perfectoid. While this construction has the useful feature of providing many examples of perfectoid fields, we will have no further need for it here. (See however Remark 1.6.5.)

1.4. Inverting the perfect norm field functor. So far, we have a functor from perfectoid analytic fields to perfect analytic fields of characteristic p . In order to invert this functor, we must also keep track of the kernel of the map Θ ; this kernel turns out to contain elements which behave a bit like linear polynomials, in that they admit an analogue of the division algorithm. This exploits an imperfect but useful analogy between strict p -rings and rings of formal power series, in which p plays the role of a series variable and the Teichmüller coordinates (Definition 1.1.5) play the role of coefficients. For a bit more on this analogy, see Remark 1.4.12; for further discussion, including a form of Weierstrass preparation in strict p -rings, see for instance [5].

Hypothesis 1.4.1. Throughout §1.4, let F be an analytic field which is perfect of characteristic p . We denote the norm on F by $|\cdot|'$.

Remark 1.4.2. We will use frequently the following consequence of the homogeneity aspect of Remark 1.1.7: the function

$$\sum_{n=0}^{\infty} p^n [\bar{x}_n] \mapsto \sup_n \{|\bar{x}_n|'\}$$

satisfies the strong triangle inequality, and hence defines a norm on $W(\mathfrak{o}_F)$. See Lemma 1.7.2 for a related observation.

Definition 1.4.3. For $z \in W(\mathfrak{o}_F)$ with reduction $\bar{z} \in \mathfrak{o}_F$, we say z is *primitive* if $|\bar{z}|' = p^{-1}$ and $p^{-1}(z - [\bar{z}]) \in W(\mathfrak{o}_F)^\times$.

Exercise 1.4.4. If $z \in W(\mathfrak{o}_F)$ is primitive, then so is uz for any $u \in W(\mathfrak{o}_F)^\times$. (Hint: this only requires a computation mod p^2 .)

In order to state the division lemma for primitive elements, we need a slightly wider class of elements of $W(\mathfrak{o}_F)$ than the Teichmüller lifts.

Definition 1.4.5. An element $x = \sum_{n=0}^{\infty} p^n [\bar{x}_n] \in W(\mathfrak{o}_F)$ is *stable* if $|\bar{x}_n|' \leq |\bar{x}_0|'$ for all $n > 0$. Note that 0 is stable under this definition.

Lemma 1.4.6. *An element of $W(\mathfrak{o}_F)$ is stable if and only if it equals a unit times a Teichmüller lift.*

Proof. This is immediate from the fact that $[\bar{x}] \sum_{n=0}^{\infty} p^n [\bar{y}_n] = \sum_{n=0}^{\infty} p^n [\bar{x}\bar{y}_n]$. □

Here is the desired analogue of the division lemma, taken from [10, Lemma 5.5].

Lemma 1.4.7. *For any primitive $z \in W(\mathfrak{o}_F)$, every class in $W(\mathfrak{o}_F)/(z)$ is represented by a stable element of $W(\mathfrak{o}_F)$.*

Proof. Write $z = [\bar{z}] + pz_1$ with $z_1 \in W(\mathfrak{o}_F)^\times$. Given $x \in W(\mathfrak{o}_F)$, put $x_0 = x$. Given $x_l = \sum_{n=0}^{\infty} p^n [\bar{x}_{l,n}]$ congruent to x modulo z , put $x_{l,1} = \sum_{n=0}^{\infty} p^n [\bar{x}_{l,n+1}]$ and $x_{l+1} = x_l - x_{l,1}z_1^{-1}z$, so that x_{l+1} is also congruent to x modulo z .

Suppose that for some l , we have $|\bar{x}_{l,n}|' < p|\bar{x}_{l,0}|'$ for all $n > 0$. By Remark 1.4.2 and the equality $x_{l+1} = [\bar{x}_{l,0}] - x_{l,1}z_1^{-1}[\bar{z}]$, we have $|\bar{x}_{l+1,n}|' \leq |\bar{x}_{l,0}|'$ for all $n \geq 0$. Also, $\bar{x}_{l+1,0}$ equals $\bar{x}_{l,0}$ plus something of lesser norm (namely $\bar{z}\bar{x}_{l,1}$ times the reduction of z_1^{-1}), so $|\bar{x}_{l+1,0}|' = |\bar{x}_{l,0}|'$. Hence x_{l+1} is a stable representative of the congruence class of x .

We may thus suppose that no such l exists. By Remark 1.4.2 again, $\sup_n \{|\bar{x}_{l+1,n}|\} \leq p^{-1} \sup_n \{|\bar{x}_{l,n}|\}$ for all l . The sum $y = \sum_{l=0}^{\infty} x_{l,1}z_1^{-1}$ thus converges for the $(p, [\bar{z}])$ -adic topology on $W(\mathfrak{o}_F)$ and satisfies $x = yz$; that is, 0 is a stable representative of the congruence class of x . □

Remark 1.4.8. One might hope that one can always take the stable representative in Lemma 1.4.7 to be a Teichmüller lift, but this is in general impossible unless the field F is not only complete but also *spherically complete*. (This condition means that any decreasing sequence of balls in F has nonempty intersection; completeness only imposes this condition when the radii of the balls tend to 0.)

Lemma 1.4.9. *Any stable element of $W(\mathfrak{o}_F)$ divisible by a primitive element must equal 0.*

Proof. Suppose $x \in W(\mathfrak{o}_F)$ is stable and is divisible by a primitive element. Put $y = x/z$ and write $x = \sum_{n=0}^{\infty} p^n [\bar{x}_n]$, $y = \sum_{n=0}^{\infty} p^n [\bar{y}_n]$. Also write $z = [\bar{z}] + pz_1$ with $z_1 \in W(\mathfrak{o}_F)^\times$; we may then write

$$y = [\bar{z}]^{-1}(x - pz_1y).$$

By induction on n and Remark 1.4.2, we deduce that $|\bar{y}_n|' = p^{n+1}|\bar{x}_0|'$; this is impossible for $y \in W(\mathfrak{o}_F)$ unless $x = 0$, as desired. \square

Corollary 1.4.10. *Suppose that $z \in W(\mathfrak{o}_F)$ is primitive and that $x, y \in W(\mathfrak{o}_F)$ are stable and congruent modulo z . Then the reductions of x, y modulo p have the same norm.*

Proof. Put $w = x - y$ and write $w = \sum_{n=0}^{\infty} p^n [\bar{w}_n]$, $x = \sum_{n=0}^{\infty} p^n [\bar{x}_n]$, $y = \sum_{n=0}^{\infty} p^n [\bar{y}_n]$. By Remark 1.4.2, $|\bar{w}_n|' \leq \max\{|\bar{x}_0|', |\bar{y}_0|'\}$ for all $n \geq 0$. However, if $|\bar{x}_0|' \neq |\bar{y}_0|'$, then $|\bar{w}_0|' = \max\{|\bar{x}_0|', |\bar{y}_0|'\} > 0$, so w is a nonzero stable element of $W(\mathfrak{o}_F)$ divisible by z . This contradicts Lemma 1.4.9, so we must have $|\bar{x}_0|' = |\bar{y}_0|'$ as desired. \square

Exercise 1.4.11. Give another proof of Lemma 1.4.9 by formulating a theory of Newton polygons for elements of $W(\mathfrak{o}_F)$.

Remark 1.4.12. A good way to understand the preceding discussion is to compare it to the theory of *Weierstrass preparation* for power series over a complete discrete valuation ring. For a concrete example, consider the ideal $(T - p)$ in the ring $\mathbb{Z}_p[[T]]$. There is a natural map $\mathbb{Z}_p \rightarrow \mathbb{Z}_p[[T]]/(T - p)$; one may see that this map is injective by observing that no nonzero element of \mathbb{Z}_p can be divisible by $T - p$ (by analogy with Lemma 1.4.9), and that it is surjective by observing that one can perform the division algorithm on power series to reduce them modulo $T - p$ to elements of \mathbb{Z}_p (by analogy with Lemma 1.4.7).

In the situation considered here, however, we do not start with a candidate for the quotient ring $W(\mathfrak{o}_F)/(z)$. Instead, we must be a bit more careful in order to read off the properties of the quotient directly from the division algorithm.

We are now ready to invert the perfect norm field functor.

Theorem 1.4.13. *Choose any primitive $z \in W(\mathfrak{o}_F)$ and put $\mathfrak{o}_K = W(\mathfrak{o}_F)/(z)$. For $x \in \mathfrak{o}_K$, apply Lemma 1.4.7 to find a stable element $y = \sum_{n=0}^{\infty} p^n [\bar{y}_n] \in W(\mathfrak{o}_F)$ lifting x , then define $|x| = |\bar{y}_0|'$. This is independent of the choice of y thanks to Lemma 1.4.9.*

- (a) *The function $|\cdot|$ is a multiplicative norm on \mathfrak{o}_K under which \mathfrak{o}_K is complete.*
- (b) *There is a natural (in F) isomorphism $\mathfrak{o}_K/(p) \cong \mathfrak{o}_F/(\bar{z})$.*
- (c) *The ring \mathfrak{o}_K is the valuation ring of an analytic field K of mixed characteristics.*
- (d) *The field K is perfectoid and there is a natural isomorphism $K' \cong F$.*
- (e) *The kernel of $\Theta : W(\mathfrak{o}_F) \rightarrow \mathfrak{o}_K$ is generated by z .*

Proof. Part (a) follows from the fact that the product of stable elements is stable (thanks to Remark 1.4.2). Part (b) follows by comparing both sides to $W(\mathfrak{o}_F)/(p, [\bar{z}])$. Part (c) follows from the fact (a consequence of Lemma 1.4.6) that if $x = \sum_{n=0}^{\infty} p^n [\bar{x}_n]$, $y = \sum_{n=0}^{\infty} p^n [\bar{y}_n]$ are stable and $|\bar{x}_n|' \leq |\bar{y}_n|'$, then x is divisible by y in $W(\mathfrak{o}_F)$. Part (d) follows from (b) plus Theorem 1.3.3. Part (e) follows from the construction of $|\cdot|$, or more precisely from the fact that every nonzero class in $W(\mathfrak{o}_F)/(z)$ has a nonzero stable representative. \square

Corollary 1.4.14. *Let K be a perfectoid analytic field. Then there exists a primitive element z in the kernel of $\Theta : W(\mathfrak{o}_{K'}) \rightarrow \mathfrak{o}_K$, so $\ker(\Theta)$ is principal generated by z by Theorem 1.4.13. (Exercise 1.4.4 then implies that conversely, any generator of $\ker(\Theta)$ is primitive.)*

Proof. Since K and K' have the same norm group by Lemma 1.3.3, we can find $\bar{z} \in \mathfrak{o}_{K'}$ with $|\bar{z}'| = p^{-1}$. Then $\theta(\bar{z})$ is divisible by p in \mathfrak{o}_K ; since Θ is surjective, we can find $z_1 \in W(\mathfrak{o}_{K'})$ with $\Theta(z_1) = -\theta(\bar{z})/p$. This forces $z_1 \in W(\mathfrak{o}_{K'})^\times$, as otherwise we would have $|\Theta(z_1)| < 1$. Now $z = [\bar{z}] + pz_1$ is a primitive element of $\ker(\Theta)$, as desired. \square

Example 1.4.15. In Example 1.3.5, note that $|\bar{\pi}'| = p^{-p/(p-1)}$. One may then check that

$$z = ([1 + \bar{\pi}] - 1)/([1 + \bar{\pi}]^{1/p} - 1) = \sum_{i=0}^{p-1} [1 + \bar{\pi}]^{i/p}$$

is a primitive element of $W(\mathfrak{o}_{K'})$ belonging to $\ker(\Theta)$. Hence z generates the kernel by Theorem 1.4.13.

Example 1.4.16. One can also write down explicit primitive elements in some cases of Example 1.3.6. A simple example is when $\pi = p$ (this forces the field F to be absolutely unramified). In this case,

$$z = p - [\bar{\pi}]$$

is a primitive element of $W(\mathfrak{o}_{K'})$ belonging to (and hence generating) $\ker(\Theta)$.

1.5. Compatibility with finite extensions. At this point, using Theorem 1.4.13 and Corollary 1.4.14, we obtain the following statement, which one might call the *perfectoid correspondence*.

Theorem 1.5.1 (Perfectoid correspondence). *The operations*

$$K \rightsquigarrow (K', \ker(\Theta : W(\mathfrak{o}_{K'}) \rightarrow \mathfrak{o}_K)), \quad (F, I) \rightsquigarrow W(\mathfrak{o}_F)[p^{-1}]/I$$

define an equivalence of categories between perfectoid analytic fields K and pairs (F, I) in which F is a perfect analytic field of characteristic p and I is an ideal of $W(\mathfrak{o}_F)$ generated by a primitive element.

Remark 1.5.2. From Example 1.3.5 and Example 1.3.6, we see that one cannot drop the ideal I in Theorem 1.4.13: one can have nonisomorphic perfectoid analytic fields whose perfect norm fields are isomorphic.

We will establish that the perfectoid correspondence is compatible with finite extensions of fields on both sides, where on the right side we replace the ideal I by its extension to the larger ring. This will give Theorem 0.0.1 by taking K to be the completion of $\mathbb{Q}_p(\mu_{p^\infty})$. However, the more general result for an arbitrary perfectoid K is relevant for extending p -adic Hodge theory to a relative setting; see Remark 1.5.8.

The first step is to lift finite extensions from characteristic p ; this turns out to be straightforward.

Lemma 1.5.3. *Let K be a perfectoid analytic field and put $I = \ker(\Theta : W(\mathfrak{o}_{K'}) \rightarrow \mathfrak{o}_K)$. Let F be a finite extension of K' and put $L = W(\mathfrak{o}_F)[p^{-1}]/IW(\mathfrak{o}_F)[p^{-1}]$. Then $[L : K] = [F : K']$. (Note that by Theorem 1.4.13, L is a perfectoid analytic field and we may identify L' with F .)*

Proof. Apply Corollary 1.4.14 to choose a primitive generator $z \in I$. The extension F/K' is separable because K' is perfect; it thus has a Galois closure \tilde{F} . Put $\tilde{L} = W(\mathfrak{o}_{\tilde{F}})[p^{-1}]/(z)$,

which is a perfectoid analytic field thanks to Theorem 1.4.13. For any subgroup H of $G = \text{Gal}(\tilde{F}/K')$, averaging over H defines a projection

$$\tilde{L} = \frac{W(\mathfrak{o}_{\tilde{F}})[p^{-1}]}{zW(\mathfrak{o}_{\tilde{F}})[p^{-1}]} \rightarrow \frac{W(\mathfrak{o}_{\tilde{F}^H})[p^{-1}]}{zW(\mathfrak{o}_{\tilde{F}})[p^{-1}] \cap W(\mathfrak{o}_{\tilde{F}^H})[p^{-1}]} = \frac{W(\mathfrak{o}_{\tilde{F}^H})[p^{-1}]}{zW(\mathfrak{o}_{\tilde{F}^H})[p^{-1}]},$$

so the right side equals the fixed field \tilde{L}^H .

Put $\tilde{G} = \text{Gal}(\tilde{F}/F)$. Apply the above analysis with $H = G$ and $H = \tilde{G}$; since $\tilde{F}^G = K'$ and $\tilde{F}^{\tilde{G}} = F$, we obtain $\tilde{L}^G = K$ and $\tilde{L}^{\tilde{G}} = L$. By Artin's lemma, \tilde{L} is a finite Galois extension of \tilde{L}^H with group H , so

$$[L : K] = [\tilde{L} : K]/[\tilde{L} : L] = \#G/\#\tilde{G} = [\tilde{F} : K']/[\tilde{F} : F] = [F : K'],$$

as desired. \square

We still need to check that a finite extension of a perfectoid analytic field is again perfectoid. It suffices to study the case where the perfect norm field is algebraically closed.

Lemma 1.5.4. *If K is a perfectoid field and K' is algebraically closed, then so is K .*

Proof. Let $P(T) \in \mathfrak{o}_K[T]$ be an arbitrary monic of degree $d \geq 1$; it suffices to check that $P(T)$ has a root in \mathfrak{o}_K . We will achieve this by exhibiting a sequence x_0, x_1, \dots of elements of \mathfrak{o}_K such that for all $n \geq 0$, $|P(x_n)| \leq p^{-n}$ and $|x_{n+1} - x_n| \leq p^{-n/d}$. This sequence will then have a limit $x \in \mathfrak{o}_K$ which is a root of P .

To begin, take $x_0 = 0$. Given $x_n \in \mathfrak{o}_K$ with $|P(x_n)| \leq p^{-n}$, write $P(T + x_n) = \sum_i Q_i T^i$. If $Q_0 = 0$, we may take $x_{n+1} = x_n$, so assume hereafter that $Q_0 \neq 0$. Put

$$c = \min\{|Q_0/Q_j|^{1/j} : j > 0, Q_j \neq 0\};$$

by taking $j = d$, we see that $c \leq |Q_0|^{1/d}$. Also, since K has the same norm group as K' by Theorem 1.4.13, this norm group is divisible; we thus have $c = |u|$ for some $u \in \mathfrak{o}_K$.

Apply Corollary 1.4.14 to construct a primitive element $z \in \ker(\Theta)$. For each i , choose $\bar{R}_i \in \mathfrak{o}_{K'}$ whose image in $\mathfrak{o}_{K'}/(\bar{z}) \cong \mathfrak{o}_K/(p)$ is the same as that of $Q_i u^i / Q_0$. Define the polynomial $\bar{R}(T) = \sum_i \bar{R}_i T^i \in \mathfrak{o}_{K'}[T]$. By construction, the largest slope in the Newton polygon of \bar{R} is 0; by this observation plus the fact that K' is algebraically closed, it follows that $\bar{R}(T)$ has a root $y' \in \mathfrak{o}_{K'}^\times$. Choose $y \in \mathfrak{o}_K^\times$ whose image in $\mathfrak{o}_K/(p) \cong \mathfrak{o}_{K'}/(\bar{z})$ is the same as that of y' , and take $x_{n+1} = x_n + uy$. Then $\sum_i Q_i u^i y^i / Q_0 \equiv 0 \pmod{p}$, so $|P(x_{n+1})| \leq p^{-1}|Q_0| \leq p^{-n-1}$ and $|x_{n+1} - x_n| = |u| \leq |Q_0|^{1/d} \leq p^{-n/d}$. We thus obtain the desired sequence, proving the claim. \square

Remark 1.5.5. The inertia subgroup of the Galois group of a finite extension of analytic fields is solvable; see for instance [9, Chapter 3] and references therein. (The discretely valued case may be more familiar; for that, see also [16, Chapter IV].) Using this statement, one can give a slightly simpler proof of Lemma 1.5.4 by considering only cyclic extensions of prime degree. We chose not to proceed this way so as to make good on our promise to keep the proof of Theorem 0.0.1 entirely free of ramification theory.

We are now ready to complete the proof of Theorem 0.0.1.

Theorem 1.5.6. *Let K be a perfectoid analytic field. Then every finite extension of K is perfectoid, and the operation $L \rightsquigarrow L'$ defines a functorial correspondence between the*

finite extensions of K and K' . In particular, the absolute Galois groups of K and K' are homeomorphic.

Proof. Apply Corollary 1.4.14 to construct a primitive generator $z \in \ker(\Theta : W(\mathfrak{o}_{K'}) \rightarrow \mathfrak{o}_K)$. Let M' be the completion of an algebraic closure \overline{K}' of K' ; it is again algebraically closed by Remark 1.2.2. By Theorem 1.5.1, M' arises as the perfect norm field of a perfectoid analytic field M , which by Lemma 1.5.4 is also algebraically closed.

By Lemma 1.5.3, each finite Galois extension L' of K' within M' is the perfect norm field of a finite Galois extension L of K within M which is perfectoid. The union \tilde{L} of such fields L is dense in M because the union of the L' is dense in M' . By Remark 1.2.2, \tilde{L} is algebraically closed; that is, every finite extension of K is contained in a finite Galois extension which is perfectoid. The rest follows from Theorem 1.5.1. \square

Remark 1.5.7. The proof of Theorem 1.5.6 is a digested version of the one given in [11]. A different proof has been given by Scholze [13], in which the analysis of strict p -rings is supplanted by use of a small amount of *almost ring theory*, as introduced by Faltings and developed systematically by Gabber and Ramero [8]. However, each of the two approaches resembles the other far more strongly than the original arguments of Fontaine and Wintenberger; it may prove useful to understand better the relationship between them.

Remark 1.5.8. In both [11] and [13], Theorem 1.5.6 is generalized to a statement relating the étale sites of certain nonarchimedean analytic spaces in characteristic 0 and characteristic p , including an optimally general form of Faltings's *almost purity theorem*. (For the flavor of this result, see Theorem 1.6.2.) This is used as a basis for relative p -adic Hodge theory in [12] and [14]. Note that for this application, it is crucial to have Theorem 1.5.6 and not just Theorem 0.0.1: one must use analytic spaces in the sense of either Berkovich or Huber rather than rigid analytic spaces, which forces an encounter with general analytic fields. (Using Huber's spaces also involves dealing with valuations of rank greater than 1, but this adds no essential difficulty.)

1.6. Some applications. We describe now a couple of applications of Theorem 1.5.6, in which one derives information in characteristic 0 by exploiting Frobenius as if one were working in positive characteristic.

Definition 1.6.1. An analytic field K is *deeply ramified* if for any finite extension L of K , $\Omega_{\mathfrak{o}_L/\mathfrak{o}_K} = 0$; that is, the morphism $\mathrm{Spec}(\mathfrak{o}_L) \rightarrow \mathrm{Spec}(\mathfrak{o}_K)$ is formally unramified. (Beware that this morphism is usually not of finite type if K is not discretely valued.)

Theorem 1.6.2. *Any perfectoid analytic field is deeply ramified.*

Proof. Let K be a perfectoid field and let L be a finite extension of K . Choose $x_1, \dots, x_n \in \mathfrak{o}_L$ which form a basis of L over K ; we can then find $t \in \mathfrak{o}_K - \{0\}$ such that $t\mathfrak{o}_L \subseteq \mathfrak{o}_K x_1 + \dots + \mathfrak{o}_K x_n$. Since L is a finite separable extension of K , $\Omega_{L/K} = 0$; consequently, we can choose $u \in \mathfrak{o}_K - \{0\}$ so that $u dx_i$ vanishes in $\Omega_{\mathfrak{o}_L/\mathfrak{o}_K}$ for each i . For any $x \in \mathfrak{o}_L$, $t dx = d(tx)$ is a \mathfrak{o}_K -linear combination of x_1, \dots, x_n , so $tu dx = 0$.

On the other hand, L is perfectoid by Theorem 1.5.6. Hence for any $x \in \mathfrak{o}_L$, we can find $y \in \mathfrak{o}_L$ for which $x \equiv y^p \pmod{p}$; this implies that $\Omega_{\mathfrak{o}_L/\mathfrak{o}_K} = p\Omega_{\mathfrak{o}_L/\mathfrak{o}_K}$. As a result, $\Omega_{\mathfrak{o}_L/\mathfrak{o}_K} = p^n \Omega_{\mathfrak{o}_L/\mathfrak{o}_K}$ for any positive integer n ; by choosing n large enough that tu is divisible by p^n , we deduce that $\Omega_{\mathfrak{o}_L/\mathfrak{o}_K} = 0$. Hence K is deeply ramified. \square

Remark 1.6.3. Theorem 1.6.2 admits the following converse: any analytic field of mixed characteristics which is deeply ramified is also perfectoid. See [8, Proposition 6.6.6].

Theorem 1.6.4. *Let K be a perfectoid analytic field and let L be a finite extension of K . Then $\text{Trace} : \mathfrak{m}_L \rightarrow \mathfrak{m}_K$ is surjective.*

Proof. By Theorem 1.5.6, L is also perfectoid. Let K', L' be the perfect norm fields of K, L . Since L' is a finite separable extension of K' , there exists $\bar{u} \in \mathfrak{m}_{K'}$ such that $\bar{u}\mathfrak{o}_{K'} \subseteq \text{Trace}(\mathfrak{m}_{L'})$. By applying the inverse of Frobenius, we obtain the same conclusion with \bar{u} replaced by $\bar{u}^{p^{-n}}$ for each positive integer n . Hence $\text{Trace} : \mathfrak{m}_{L'} \rightarrow \mathfrak{m}_{K'}$ is surjective.

Since K is not discretely valued, we can find $t \in \mathfrak{m}_K$ with $p^{-1} < |t| < 1$. Since L is a finite separable extension of K , there exists a nonnegative integer m such that $(p/t)^m \mathfrak{m}_K \subseteq \text{Trace}(\mathfrak{m}_L)$. If $m > 0$, then for each $x \in (p/t)^{m-1} \mathfrak{m}_K$, by the previous paragraph we may write $x = \text{Trace}(y) + pz$ for some $y \in \mathfrak{m}_L, z \in \mathfrak{o}_K$; since $pz = (p/t)(tz) \in (p/t)^m \mathfrak{m}_K$, it follows that $z \in \text{Trace}(\mathfrak{m}_L)$ and hence $x \in \text{Trace}(\mathfrak{m}_L)$. In other words, we may replace m by $m - 1$; this proves the desired result. \square

Remark 1.6.5. Note that Theorem 1.6.4 still holds, with the same proof, if we replace the perfectoid field K by a dense subfield as long as the norm extends uniquely to the finite extension L of K . For instance, we may take K to be an infinite algebraic extension of an analytic field F of mixed characteristics.

One important case is when $F = \mathbb{Q}_p$ and K is a Galois extension whose Galois group contains \mathbb{Z}_p (e.g., any p -adic Lie group). In this case, the perfectoid property can be checked using a study of higher ramification groups; the technique was introduced by Tate and developed further by Sen [15] (see also [3, §13]).

However, no ramification theory is necessary in case K is a field for which the perfectoid property can be checked directly (as in Example 1.3.5 or Example 1.3.6), or a finite extension of such a field (using Theorem 1.5.6).

1.7. Gauss norms. We conclude this section by appending some more observations about norms on strict p -rings, for use in the second half of the paper.

Hypothesis 1.7.1. Throughout §1.7, again let F be an analytic field which is perfect of characteristic p , with norm $|\cdot|'$.

The key construction here is an analogue of the Gauss norm, following [10, Lemma 4.1].

Lemma 1.7.2. *For $r > 0$, the formula*

$$(1.7.2.1) \quad \left| \sum_{n=0}^{\infty} p^n [\bar{x}_n] \right|_r = \sup_n \{ p^{-n} (|\bar{x}_n|')^r \}$$

defines a function $|\cdot|_r : W(F) \rightarrow [0, +\infty]$ satisfying the strong triangle inequality $|x + y|_r \leq \max\{|x|_r, |y|_r\}$ and the multiplicativity property $|xy|_r = |x|_r |y|_r$ for all $x, y \in W(F)$ (under the convention $0 \times +\infty = 0$). In particular, the subset $W^r(F)$ of $W(F)$ on which $|\cdot|_r$ is finite forms a ring on which $|\cdot|_r$ defines a multiplicative nonarchimedean norm.

Proof. From Remark 1.1.7, it follows that $|\bar{x} + \bar{y}|_r \leq \max\{|\bar{x}|_r, |\bar{y}|_r\}$ for all $\bar{x}, \bar{y} \in \mathfrak{o}_F$ (as in Remark 1.4.2). It follows easily that for $x, y \in F$, $|x + y|_r \leq \max\{|x|_r, |y|_r\}$ and

$|xy|_r \leq |x|_r |y|_r$. To establish multiplicativity, by continuity it is enough to consider finite sums $x = \sum_{m=0}^M p^m [\bar{x}_m]$, $y = \sum_{n=0}^N p^n [\bar{y}_n]$. For all but finitely many $r > 0$, the quantities

$$p^{-m-n} (|\bar{x}_m \bar{y}_n|')^r \quad (m = 0, \dots, M; n = 0, \dots, N)$$

are all distinct; for such r , the fact that $|\cdot|_r$ is a nonarchimedean norm (shown above) gives

$$|xy|_r = \max\{p^{-m-n} (|\bar{x}_m \bar{y}_n|')^r : m = 0, \dots, M; n = 0, \dots, N\} = |x|_r |y|_r.$$

Since $|xy|_r$ is a continuous function of r , we may infer the claim also in the exceptional cases. \square

Remark 1.7.3. For $s \in (0, r]$ and $x = \sum_{n=0}^{\infty} p^n [\bar{x}_n]$,

$$(1.7.3.1) \quad |x|_s = \sup_n \{p^{-n} (|\bar{x}_n|')^s\} \leq \sup_n \{p^{-ns/r} (|\bar{x}_n|')^s\} = |x|_r^{s/r}.$$

An even stronger statement is that for $x \in W(F)$ fixed, the function $\log(|\cdot|_r) : W(F) \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex (because it is the supremum of convex functions); this bears a certain formal resemblance to the *Hadamard three circles theorem* in complex analysis.

Definition 1.7.4. Write $W^\dagger(F)$ for the union $\cup_{r>0} W^r(F)$. This is a local ring which is not complete, but can be easily shown to be *henselian*; that is, every finite étale F -algebra lifts uniquely to a finite étale $W^\dagger(F)$ -algebra.

Exercise 1.7.5. One often works with the following variant of the construction of $W^r(F)$. (In fact, it is often this ring which is defined as $W^r(F)$, rather than the one defined in Lemma 1.7.2; however, statement (c) below shows that this change does not alter the definition of $W^\dagger(F)$.)

- (a) For any $\bar{z} \in \mathfrak{o}_F$ with $|\bar{z}|' < 1$, the ring $W(\mathfrak{o}_F)[[\bar{z}]^{-1}]$ consists of those elements of $W(F)$ with bounded Teichmüller coordinates.
- (b) The completion of $W(\mathfrak{o}_F)[[\bar{z}]^{-1}]$ under $|\cdot|_r$ consists of those $x = \sum_{n=0}^{\infty} p^n [\bar{x}_n]$ for which $\lim_{n \rightarrow \infty} p^{-n} (|\bar{x}_n|')^r = 0$. In particular, these form a ring.
- (c) The ring defined in (b) is contained in $W^r(F)$ and contains $W^s(F)$ for all $s > r$.

Remark 1.7.6. Let φ denote the endomorphism of $W(F)$ induced by the Frobenius map on F . For any $r > 0$, using the fact that $W^\dagger(F) = \cup_{n=0}^{\infty} W^{p^{-n}r}(F) = \cup_{n=0}^{\infty} \varphi^{-n}(W^r(F))$, it can be shown that every finite étale F -algebra lifts uniquely to a finite étale $W^r(F)$ -algebra. In case $F = K'$ for some deeply ramified analytic field K of mixed characteristics, the map $\Theta : W(\mathfrak{o}_F) \rightarrow \mathfrak{o}_K$ extends to a homomorphism $W^r(F) \rightarrow K$ for any $r > 1$, and Theorem 1.5.6 is equivalent to the statement that every finite étale K -algebra lifts uniquely to $W^r(F)$. (If we change the definition of $W^r(F)$ as per Exercise 1.7.5, we may also take $r = 1$ in this last statement.)

2. GALOIS REPRESENTATIONS AND (φ, Γ) -MODULES

Hypothesis 2.0.1. Throughout §2, let K denote a finite extension of \mathbb{Q}_p , and write G_K for the absolute Galois group of K .

Convention 2.0.2. When working with a matrix A over a ring carrying a norm $|\cdot|$, we will write $|A|$ for the supremum of the norms of the entries of A (rather than the operator norm or spectral radius).

Definition 2.0.3. Let $\mathbf{F\acute{E}t}(R)$ denote the category of finite étale algebras over a ring R . For R a field, these are just the direct sums of finite separable field extensions of R .

2.1. Some period rings over \mathbb{Q}_p . We will consider four different rings which can classify G_K -representations. We first introduce them all in the case $K = \mathbb{Q}_p$.

Definition 2.1.1. Let L be the completed perfect closure of $\mathbb{F}_p((\bar{\pi}))$, and put $\tilde{\mathbf{A}}_{\mathbb{Q}_p} = W(L)$. This defines a complete topological ring both for the p -adic topology and for the *weak topology*, under which a sequence converges if each sequence of Teichmüller coordinates converges in the norm topology on L . (The restriction of the weak topology to $W(\mathfrak{o}_L)$ coincides with the $(p, [\bar{\pi}]$)-adic topology.) Let φ be the endomorphism of $\tilde{\mathbf{A}}_{\mathbb{Q}_p}$ induced by the Frobenius map on L .

Definition 2.1.2. Let $\mathbf{A}_{\mathbb{Q}_p}$ be the p -adic completion of $\mathbb{Z}((\pi))$; it is a Cohen ring (a complete discrete valuation ring with maximal ideal (p)) with $\mathbf{A}_{\mathbb{Q}_p}/(p) \cong \mathbb{F}_p((\bar{\pi}))$. We identify $\mathbf{A}_{\mathbb{Q}_p}$ with a subring of $\tilde{\mathbf{A}}_{\mathbb{Q}_p}$ in such a way that π corresponds to $[1 + \bar{\pi}] - 1$. Note that φ then acts on $\mathbf{A}_{\mathbb{Q}_p}$ as the \mathbb{Z}_p -linear substitution $\pi \mapsto (1 + \pi)^p - 1$, and a sequence in $\mathbf{A}_{\mathbb{Q}_p}$ converges for the weak topology if and only if its image in $\mathbf{A}_{\mathbb{Q}_p}/(p^n)$ converges π -adically for each positive integer n .

Definition 2.1.3. Put $\tilde{\mathbf{A}}_{\mathbb{Q}_p}^\dagger = W^\dagger(L) = \cup_{r>0} W^r(L)$. Since each $W^r(L)$ is complete for the maximum of $|\cdot|_r$ and the p -adic norm, $\tilde{\mathbf{A}}_{\mathbb{Q}_p}^\dagger$ is an incomplete but henselian local ring contained in $W(L) = \tilde{\mathbf{A}}_{\mathbb{Q}_p}$. Note that φ acts bijectively on $\tilde{\mathbf{A}}_{\mathbb{Q}_p}^\dagger$. We equip $\tilde{\mathbf{A}}_{\mathbb{Q}_p}^\dagger$ with the p -adic and weak topologies by restriction from $\tilde{\mathbf{A}}_{\mathbb{Q}_p}$; we also define the *LF topology*, in which a sequence converges if and only if it converges in some $W^r(L)$. (LF is an abbreviation for *limit of Fréchet*.)

Definition 2.1.4. Put $\mathbf{A}_{\mathbb{Q}_p}^\dagger = \tilde{\mathbf{A}}_{\mathbb{Q}_p}^\dagger \cap \mathbf{A}_{\mathbb{Q}_p}$; since $\mathbb{Z}_p[\pi^\pm] \subset \mathbf{A}_{\mathbb{Q}_p}^\dagger$, $\mathbf{A}_{\mathbb{Q}_p}^\dagger$ is again a henselian local ring with residue field $\mathbb{F}_p((\bar{\pi}))$ on which φ acts. It inherits p -adic, weak, and LF topologies. For a more concrete description of $\mathbf{A}_{\mathbb{Q}_p}^\dagger$, see Corollary 2.2.9.

Definition 2.1.5. For $\gamma \in \Gamma = \mathbb{Z}_p^\times$, let $\gamma : \mathbf{A}_{\mathbb{Q}_p} \rightarrow \mathbf{A}_{\mathbb{Q}_p}$ be the \mathbb{Z}_p -linear substitution $\pi \mapsto (1 + \pi)^\gamma - 1$, where $(1 + \pi)^\gamma$ is defined by its binomial expansion. The induced map on $\mathbb{F}_p((\bar{\pi}))$ extends to L and thus defines an action of Γ on $\tilde{\mathbf{A}}_{\mathbb{Q}_p}$; Γ also acts on $\tilde{\mathbf{A}}_{\mathbb{Q}_p}^\dagger$ and $\mathbf{A}_{\mathbb{Q}_p}^\dagger$. For $* \in \{\tilde{\mathbf{A}}, \mathbf{A}, \tilde{\mathbf{A}}^\dagger, \mathbf{A}^\dagger\}$, the action of Γ on $*_{\mathbb{Q}_p}$ is continuous (meaning that the map $\Gamma \times *_{\mathbb{Q}_p} \rightarrow *_{\mathbb{Q}_p}$ is continuous) for the weak topology and (when available) the LF topology.

Exercise 2.1.6. In Definition 2.1.5, the action of Γ on $*_{\mathbb{Q}_p}$ is not continuous for the p -adic topology, even though the action of each individual element of Γ is continuous.

2.2. Extensions of \mathbb{Q}_p . We extend the definition of the period rings to finite extensions of \mathbb{Q}_p using a refinement of Theorem 1.5.6.

Theorem 2.2.1. For $* \in \{\tilde{\mathbf{A}}, \mathbf{A}, \tilde{\mathbf{A}}^\dagger, \mathbf{A}^\dagger\}$, the category $\mathbf{F\acute{E}t}(\mathbb{Q}_p)$ is equivalent to the category of finite étale algebras over $*_{\mathbb{Q}_p}$ admitting an extension of the action of Γ . Moreover, this equivalence is compatible with the base extensions among different choices of $*$.

Proof. Put $L = \tilde{\mathbf{A}}_{\mathbb{Q}_p}/(p)$. Via Remark 1.2.3, Theorem 1.5.6, and the fact that the local rings $\tilde{\mathbf{A}}_{\mathbb{Q}_p}^\dagger$ and $\mathbf{A}_{\mathbb{Q}_p}^\dagger$ are both henselian, we see that the categories

$$\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathbb{Q}_p(\mu_{p^\infty})), \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(L), \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathbb{F}_p((\bar{\pi}))), \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\tilde{\mathbf{A}}_{\mathbb{Q}_p}), \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathbf{A}_{\mathbb{Q}_p}), \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\tilde{\mathbf{A}}_{\mathbb{Q}_p}^\dagger), \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathbf{A}_{\mathbb{Q}_p}^\dagger)$$

are all equivalent, compatibly with base extensions among different choices of $*$. It thus suffices to consider $* = \tilde{\mathbf{A}}$ in what follows.

From the explicit description given in Example 1.3.5, we see that the map $\Theta : W(\mathfrak{o}_L) \rightarrow \mathbb{Z}_p[\mu_{p^\infty}]$ becomes Γ -equivariant if we identify Γ with $\text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ via the cyclotomic character. Consequently, for $K \in \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathbb{Q}_p)$, the object in $\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\tilde{\mathbf{A}}_{\mathbb{Q}_p})$ corresponding to $K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})$ carries an action of Γ .

Conversely, suppose $S \in \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\tilde{\mathbf{A}}_{\mathbb{Q}_p})$ carries an action of Γ ; then the corresponding object E of $\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathbb{Q}_p(\mu_{p^\infty}))$ also carries an action of Γ . We may realize E as the base extension of a finite étale algebra E_n over $\mathbb{Q}_p(\mu_{p^n})$ for some nonnegative integer n ; by Artin's lemma, E_n is fixed by a subgroup of Γ of finite index, which is necessarily open. By Galois descent, E_n descends to a finite étale algebra E over \mathbb{Q}_p , as desired. \square

Definition 2.2.2. Let $\tilde{\mathbf{A}}_K, \mathbf{A}_K, \tilde{\mathbf{A}}_K^\dagger, \mathbf{A}_K^\dagger$ be the objects corresponding to K via Theorem 2.2.1. We may write $\tilde{\mathbf{A}}_K = \bigoplus W(\tilde{L}), \tilde{\mathbf{A}}_K^\dagger = \bigoplus W^\dagger(\tilde{L})$ for \tilde{L} running over the connected components of $\tilde{\mathbf{A}}_K/(p)$ (which correspond to the connected components of $K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})$). We may thus equip $*_K$ with a p -adic topology, a weak topology, and (for $* = \tilde{\mathbf{A}}^\dagger, \mathbf{A}^\dagger$) also an LF topology. Define the norm $|\cdot|'$ on $\tilde{\mathbf{A}}_K/(p)$ as the supremum over connected components; for $r > 0$, define $|\cdot|_r$ on $\tilde{\mathbf{A}}_K$ as the supremum over connected components.

The actions of φ, Γ extend to $*_K$; the action of Γ is again continuous for the weak topology and (when available) the LF topology. Note that Γ acts transitively on the \tilde{L} .

Exercise 2.2.3. Each of the topologies on $*_K$ coincides with the one obtained by viewing $*_K$ as a finite free module over $*_{\mathbb{Q}_p}$ and equipping the latter with the corresponding topology.

Remark 2.2.4. Each connected component L of $\mathbf{A}_K/(p)$ is a finite separable extension of $\mathbb{F}_p((\bar{\pi}))$, and hence is itself isomorphic to a power series field in some variable $\bar{\pi}_L$ over some finite extension \mathbb{F}_q of \mathbb{F}_p . In general, there is no distinguished choice of $\bar{\pi}_L$. One has similar (and similarly undistinguished) descriptions of \mathbf{A}_K and \mathbf{A}_K^\dagger ; see Lemma 2.2.6.

Exercise 2.2.5. In Remark 2.2.4, \mathbb{F}_q coincides with the residue field of $K(\mu_{p^\infty})$. Note that this may not equal the residue field of K .

Lemma 2.2.6. *Keep notation as in Remark 2.2.4. Let R be the connected component of \mathbf{A}_K with $R/(p) = L$, and choose $\pi_L \in R$ lifting $\bar{\pi}_L$. Then R is isomorphic to the p -adic completion of $W(\mathbb{F}_q)((\pi_L))$.*

Proof. It suffices to observe that the latter ring is indeed a finite étale algebra over $\mathbf{A}_{\mathbb{Q}_p}$ whose reduction modulo p is isomorphic to L . \square

Exercise 2.2.7. In Lemma 2.2.6, the weak topology on R (obtained by restriction from \mathbf{A}_K) coincides with the weak topology on the p -adic completion of $W(\mathbb{F}_q)((\pi_L))$ (in which as in Definition 2.1.2, a sequence converges if and only if it converges π_L -adically modulo each power of p).

Lemma 2.2.8. *Keep notation as in Lemma 2.2.6, but assume further that $\pi_L \in R^\dagger$ for R^\dagger the connected component of \mathbf{A}_K^\dagger with $R^\dagger/(p) = L$. Then there exists $r_0 > 0$ (depending on L and π_L) with the following properties.*

- (a) *Every $\bar{x} \in R/(p)$ admits a lift $x \in R$ with $|x - [\bar{x}]|_r < |x|_r$ for all $r \in (0, r_0]$.*
- (b) *For $r \in (0, r_0]$, for $x = \sum_{n \in \mathbb{Z}} x_n \pi_L^n \in R$ with $x_n \in W(\mathbb{F}_q)$,*

$$(2.2.8.1) \quad |x|_r = \sup_n \{|x_n| (|\pi_L|')^{nr}\}.$$

Proof. We have $R^\dagger = R \cap W^\dagger(L)$ because both sides are subrings of R which are finite étale over $\mathbf{A}_{\mathbb{Q}_p}^\dagger$ and which surject onto $R/(p)$. Consequently, $\pi_L \in W^r(L)$ for some $r > 0$. Since $\limsup_{r \rightarrow 0^+} |[\pi_L]|_r = 1$ while $\limsup_{r \rightarrow 0^+} |\pi_L - [\pi_L]|_r \leq p^{-1}$, we can choose $r_0 > 0$ so that

$$(2.2.8.2) \quad |\pi_L - [\pi_L]|_r < |\pi_L|_r = |[\pi_L]|_r \quad (r \in (0, r_0]).$$

We prove the claims for any such r_0 .

To prove (a), lift $\bar{x} = \sum_{n \in \mathbb{Z}} \bar{x}_n \pi_L^n \in R/(p)$ to $x = \sum_{n \in \mathbb{Z}} [\bar{x}_n] \pi_L^n \in R$. This lift satisfies $|x - [\bar{x}]|_r < (|\pi_L|')^r = |[\bar{x}]|_r = |x|_r$ for all $r \in (0, r_0]$ thanks to (2.2.8.2).

To prove (b), first note that since $|\cdot|_r$ is a norm, (2.2.8.2) implies

$$|x|_r \leq \sup_n \{|x_n| (|\pi_L|')^{nr}\}.$$

To finish, it is enough to establish by induction that for each nonnegative integer m , $|x|_r$ is at least the supremum of $|x_n| (|\pi_L|')^{nr}$ over indices n for which x_n is not divisible by p^{m+1} . This is clear for $m = 0$. If $m > 0$, there is nothing to check unless the supremum is only achieved in cases when x_n is divisible by p^m . In that case, lift the reduction \bar{x} of x modulo p as in (a) to some y with $|y - [\bar{x}]|_r < |y|_r$ for $r \in (0, r_0]$. For $z = (x - y)/p = \sum_n z_n \pi_L^n$, the supremum in question is also the supremum of $p^{-1} |z_n| (|\pi_L|')^{nr}$ over indices n for which z_n is not divisible by p^m . By the induction hypothesis, this is at most

$$p^{-1} |z|_r = |pz|_r \leq \max\{|x|_r, |y|_r\} = |x|_r.$$

This completes the induction, yielding (b). □

This gives us a concrete description of R^\dagger in terms of the coefficients of a series representation.

Corollary 2.2.9. *With notation as in Lemma 2.2.8, $x \in R^\dagger$ if and only if there exists $r > 0$ such that $\sup_n \{|x_n| (|\pi_L|')^{nr}\} < +\infty$.*

Corollary 2.2.10. *There exists $r_0 > 0$ (depending on K) such that every $\bar{x} \in \mathbf{A}_K/(p)$ admits a lift $x \in \mathbf{A}_K$ with $|x - [\bar{x}]|_r < |x|_r$ for all $r \in (0, r_0]$.*

Proof. Apply Lemma 2.2.8(a) to each connected component of \mathbf{A}_K . □

Exercise 2.2.11. With notation as in Lemma 2.2.8, one can choose r_0 so that for $r \in (0, r_0]$, $x \in W^r(L)$ if and only if $\sup_n \{|x_n| (|\pi_L|')^{nr}\} < +\infty$.

Exercise 2.2.12. Compute the optimal constant r_0 in Lemma 2.2.8 for $K = \mathbb{Q}_p$ and $\pi_L = \pi$.

Remark 2.2.13. Beware that our notations do not agree with [3] or most other references when $K \neq \mathbb{Q}_p$. It is more customary to take \mathbf{A}_K to be the finite étale algebra over $\mathbf{A}_{\mathbb{Q}_p}$ corresponding to a connected component of $K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})$, and similarly for the other period rings. These rings inherit an action of φ and of the subgroup Γ_K of Γ fixing a component of

K_∞ ; the action of Γ on \mathbf{A}_K acts transitively on connected components. Our point of view has the mild advantage of making the relationship between K and \mathbf{A}_K more uniform; for instance, for L a finite Galois extension of K , $\text{Gal}(L/K)$ acts on \mathbf{A}_L with fixed ring \mathbf{A}_K . We leave to the reader the easy task of translating back and forth between statements in terms of the usual rings and the corresponding statements in our language.

2.3. Étale (φ, Γ) -modules.

Definition 2.3.1. Let R be any of $\tilde{\mathbf{A}}_K, \mathbf{A}_K, \tilde{\mathbf{A}}_K^\dagger, \mathbf{A}_K^\dagger$. Let M be a finite free R -module. A *semilinear action* of φ on M is an additive map $\varphi : M \rightarrow M$ for which $\varphi(rm) = \varphi(r)\varphi(m)$ for all $r \in R, m \in M$. Such an action is *étale* if it takes some basis of M to another basis; the same is then true of any basis (by Remark 2.3.2 or Remark 2.3.3). An *étale φ -module* over R is a finite free R -module M equipped with an étale semilinear action of φ .

We similarly define semilinear actions of $\gamma \in \Gamma$. To define a semilinear action of Γ as a whole, we insist that the actions of individual elements compose: for all $\gamma_1, \gamma_2 \in \Gamma$ and $m \in M$, we must have $\gamma_1(\gamma_2(m)) = (\gamma_1\gamma_2)(m)$. We say an action of Γ is *continuous* if the action map $\Gamma \times M \rightarrow M$ is continuous for the weak topology and (when available) the LF topology. An *étale (φ, Γ) -module* over R is an étale φ -module M equipped with a continuous action of Γ commuting with φ . (The continuity condition can be omitted; see Exercise 2.4.6.)

Remark 2.3.2. Define $\varphi^*M = R \otimes_R M$ as a left R -module where the left tensorand R is viewed as a left R -module in the usual way and as a right R -module via φ ; that is, we have $1 \otimes rm = \varphi(r) \otimes m$ and $r(s \otimes m) = rs \otimes m$. One may then view a semilinear action of φ as an R -linear map $\Phi : \varphi^*M \rightarrow M$; the action is étale if and only if Φ is an isomorphism.

Remark 2.3.3. A semilinear φ -action can be specified in terms of a basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ by exhibiting the matrix A for which $\varphi(\mathbf{e}_j) = \sum_i A_{ij} \mathbf{e}_i$ (which we sometimes call the *matrix of action* of φ on the basis). If $\mathbf{e}'_1, \dots, \mathbf{e}'_d$ is another basis, then there exists an invertible matrix U with $\mathbf{e}'_j = \sum_i U_{ij} \mathbf{e}_i$, and the matrix of action of φ on the new basis is $U^{-1}A\varphi(U)$.

Lemma 2.3.4. *Let M be an étale (φ, Γ) -module over $\tilde{\mathbf{A}}_K$. Then for each positive integer n , there exists a finite extension L of K for which*

$$(M \otimes_{\tilde{\mathbf{A}}_K} \tilde{\mathbf{A}}_L / (p^n))^{\varphi, \Gamma} \otimes_{\mathbb{Z}_p} \tilde{\mathbf{A}}_L \rightarrow M \otimes_{\tilde{\mathbf{A}}_K} \tilde{\mathbf{A}}_L / (p^n)$$

is an isomorphism.

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be a basis of M , and define $A \in \text{GL}_d(\tilde{\mathbf{A}}_K)$ by $\varphi(\mathbf{e}_j) = \sum_i A_{ij} \mathbf{e}_i$. For each $\gamma \in \Gamma$, define $G_\gamma \in \text{GL}_d(\tilde{\mathbf{A}}_K)$ by $\gamma(\mathbf{e}_j) = \sum_i G_{\gamma, ij} \mathbf{e}_i$; then the fact that $\varphi \circ \gamma = \gamma \circ \varphi$ implies that $A\varphi(G_\gamma) = G_\gamma\gamma(A)$. Put

$$\begin{aligned} R_n &= (\tilde{\mathbf{A}}_K / (p)) [X_{ij, k}^{p^{-\infty}} : i, j = 1, \dots, d; k = 0, \dots, n-1] \\ S_n &= W(R_n) / (p^n, \varphi^m(A\varphi(X) - X) \ (m \in \mathbb{Z})), \end{aligned}$$

where X denotes the matrix with $X_{ij} = \sum_{k=0}^{n-1} p^k [X_{ij, k}]$. Note that S_n carries an action of Γ with $\gamma \in \Gamma$ sending X to $G_\gamma\gamma(X)$.

It can be shown that S_n is finite étale over $\tilde{\mathbf{A}}_K / (p^n)$. In the case $n = 1$, this reduces to observing that

$$\tilde{\mathbf{A}}_K / (p) [X_{ij} : i, j = 1, \dots, d] / (A\varphi(X) - X)$$

is étale because the derivative of φ is 0. For $n > 1$, an induction argument reduces one to checking that Artin-Schreier equations define finite étale algebras in characteristic p .

Thus via Theorem 2.2.1, S_n corresponds to a finite étale algebra over K , any connected component of which has the desired effect. \square

Theorem 2.3.5. *The following categories are equivalent.*

- (a) *The category of continuous representations of G_K on finite free \mathbb{Z}_p -modules.*
- (b) *The category of étale (φ, Γ) -modules over $\tilde{\mathbf{A}}_K$.*
- (c) *The category of étale (φ, Γ) -modules over \mathbf{A}_K .*

More precisely, the functor from (c) to (b) is base extension.

Proof. The functor from (a) to (c) is defined as follows. Let T be a finite free \mathbb{Z}_p -module, and let $\tau : G_K \rightarrow \mathrm{GL}(T)$ be a continuous homomorphism. For each positive integer n , the map $G_K \rightarrow \mathrm{GL}(T/p^n T)$ factors through $G_{L/K}$ for some finite Galois extension L of K . Put

$$M_n = (T \otimes_{\mathbb{Z}_p} \mathbf{A}_L / (p^n))^{G_{L/K}};$$

then φ and Γ act on $\mathbf{A}_L / (p^n)$ and hence on M_n , and faithfully flat descent for modules (or a more elementary Galois descent) implies that

$$(2.3.5.1) \quad M_n \otimes_{\mathbf{A}_K} \mathbf{A}_L \rightarrow T \otimes_{\mathbb{Z}_p} \mathbf{A}_L / (p^n)$$

is an isomorphism. Hence $M = \varprojlim M_n$ is an étale (φ, Γ) -module over \mathbf{A}_K .

The functor from (b) to (a) is defined as follows. Let M be an étale (φ, Γ) -module over $\tilde{\mathbf{A}}_K$. For each positive integer n , choose a finite Galois extension L of K as in Lemma 2.3.4 so that

$$T_n = (M \otimes_{\tilde{\mathbf{A}}_K} \tilde{\mathbf{A}}_L / (p^n))^{\varphi, \Gamma}$$

has the property that the natural map

$$(2.3.5.2) \quad T_n \otimes_{\mathbb{Z}_p} \tilde{\mathbf{A}}_L \rightarrow M \otimes_{\tilde{\mathbf{A}}_K} \tilde{\mathbf{A}}_L / (p^n)$$

is an isomorphism. Note that $G_{L/K}$ acts on $\tilde{\mathbf{A}}_L$ and hence on T_n ; hence $T = \varprojlim T_n$ is a continuous representation of G_K .

Using the fact that (2.3.5.1) and (2.3.5.2) are isomorphisms, it is straightforward to check that composing around the circle always gives a functor naturally isomorphic to the identity functor at the starting point. This completes the proof. \square

2.4. Overconvergence and (φ, Γ) -modules, part 1.

Remark 2.4.1. For $* \in \{\tilde{\mathbf{A}}, \mathbf{A}, \tilde{\mathbf{A}}^\dagger, \mathbf{A}^\dagger\}$, for two étale φ -modules (resp. étale (φ, Γ) -modules) over $*_K$, we may view $\mathrm{Hom}_{*_K}(M, N)$ naturally as an étale φ -module (resp. an étale (φ, Γ) -module) over $*_K$ by imposing the conditions that

$$\varphi(f)(\varphi(\mathbf{e})) = \varphi(f(\mathbf{e})), \quad \gamma(f)(\gamma(\mathbf{e})) = \gamma(f(\mathbf{e})) \quad (\gamma \in \Gamma, f \in \mathrm{Hom}_{*_K}(M, N), \mathbf{e} \in M).$$

The morphisms $M \rightarrow N$ of φ -modules (resp. of (φ, Γ) -modules) then are precisely the elements of $\mathrm{Hom}_{*_K}(M, N)$ fixed by φ (resp. fixed by φ and Γ).

Lemma 2.4.2. *Base extension of étale φ -modules which are trivial modulo p from $\tilde{\mathbf{A}}_K^\dagger$ to $\tilde{\mathbf{A}}_K$ is fully faithful.*

Proof. By Remark 2.4.1, this reduces to checking that if M is an étale φ -module over $\tilde{\mathbf{A}}_K^\dagger$ which is trivial modulo p , then

$$M^\varphi = (M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K)^\varphi.$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be a basis of M which is fixed modulo p . Let L be a connected component of $\tilde{\mathbf{A}}_K/(p)$. Using Theorem 2.3.5, we can produce an analytic field L' with an isometric embedding $L \hookrightarrow L'$ for which $M \otimes_{\tilde{\mathbf{A}}_K^\dagger} W(L)$ admits a φ -invariant basis $\mathbf{e}'_1, \dots, \mathbf{e}'_d$.

We now calculate as in [11, Proposition 5.4.5]. Let $A \in \mathrm{GL}_d(\tilde{\mathbf{A}}_K^\dagger)$ be given by $\varphi(\mathbf{e}_j) = \sum_i A_{ij} \mathbf{e}_i$; then $A - 1$ is divisible by p . Consequently, $\limsup_{r \rightarrow 0^+} |A - 1|_r \leq p^{-1}$, so we can choose $r > 0$ so that $|A - 1|_r < p^{-1/2}$. Let $U \in \mathrm{GL}_d(W(L))$ be given by $\mathbf{e}'_j = \sum_i U_{ij} \mathbf{e}_i$. We claim that for each positive integer n , U is congruent modulo p^n to some $V_n \in \mathrm{GL}_d(W^{pr}(L'))$ with $|V_n - 1|_r, |V_n - 1|_{pr} \leq p^{-1/2}$. This is clear for $n = 1$ by taking $V_n = 1$. Given the claim for some n , U is congruent modulo p^{n+1} to a matrix $V_n + p^n X$ in which each entry X_{ij} is a Teichmüller lift. We have

$$\varphi(X) - X \equiv p^{-n}(V_n - \varphi(V_n) - (A - 1)\varphi(V_n)) \pmod{p},$$

which implies that $|\varphi(X_{ij}) - X_{ij}|_r \leq p^{n-1/2}$. From this it follows in turn that

$$|X_{ij}|_{rp} \leq p^{n-1/2}.$$

We may then take $V_{n+1} = V_n + p^n X$ for the desired effect.

From the previous paragraph, it follows that $\mathbf{e}'_1, \dots, \mathbf{e}'_d$ form a basis of $M \otimes_{\tilde{\mathbf{A}}_K^\dagger} W^\dagger(L')$. By expressing a φ -invariant element of M using this basis, we see that the image of $(M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K)^\varphi$ in $M \otimes_{\tilde{\mathbf{A}}_K^\dagger} W(L')$ is contained in $M \otimes_{\tilde{\mathbf{A}}_K^\dagger} W^\dagger(L')$. Because $W(L) \cap W^\dagger(L') = W^\dagger(L)$ and M is a free module, we may further conclude that the image lands in $M \otimes_{\tilde{\mathbf{A}}_K^\dagger} W^\dagger(L)$. Since this is true for each L , we deduce that $(M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K)^\varphi \subseteq M$, as desired. \square

Lemma 2.4.3. *Let M^\dagger be an étale φ -module over $\tilde{\mathbf{A}}_K^\dagger$ which is trivial modulo p , and suppose that $M^\dagger \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K$ carries a semilinear action of Γ which commutes with φ and is continuous for the weak topology. Then the action of Γ on M^\dagger provided by Lemma 2.4.2 is continuous for the LF topology.*

Proof. Retain notation as in the proof of Lemma 2.4.2. Note that Γ acts on $M^\dagger \otimes_{\tilde{\mathbf{A}}_K^\dagger} W(L')$ continuously for the weak topology. It thus acts on $(M^\dagger \otimes_{\tilde{\mathbf{A}}_K^\dagger} W(L'))^\varphi = (M^\dagger \otimes_{\tilde{\mathbf{A}}_K^\dagger} W^\dagger(L'))^\varphi$ continuously for the weak topology. However, the latter is a finite free \mathbb{Z}_p -module on which the weak and LF topologies coincide. We thus obtain a continuous action of Γ on $M^\dagger \otimes_{\tilde{\mathbf{A}}_K^\dagger} W^\dagger(L')$ for the LF topology. Since this is true for each L , the action of Γ on M^\dagger is continuous for the LF topology, as desired. \square

Lemma 2.4.4. *Let L be an analytic field which is perfect of characteristic p , with norm $|\cdot|'$. Let M be a finite free $W(L)$ -module equipped with an étale semilinear φ -action and admitting a basis which is fixed by φ modulo p . Then there exists a basis of $W(L)$ on which φ acts via an invertible matrix over $W^\dagger(L)$.*

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be a basis of M which is fixed modulo p , and let $F \in \mathrm{GL}_d(W(L))$ be defined by $\varphi(\mathbf{e}_j) = \sum_i F_{ij} \mathbf{e}_i$. We construct sequences of matrices F_n, G_n such that

$F_1 = F$, $G_1 = 1$, $F_n - 1$ has entries in $pW(L)$, G_n has entries in $W^1(L)$, $|G_n - 1|_1 < 1$, and $X_n = p^{-n}(F_n - G_n)$ has entries in $W(L)$. Given F_n and G_n , choose a nonnegative integer m for which $|\varphi^{-m}(\overline{X}_n)|' < p^{n/2}$, put $\overline{Y}_n = -\sum_{h=1}^m \varphi^{-h}(\overline{X}_n)$, and put

$$U_n = 1 + p^n[\overline{Y}_n], \quad F_{n+1} = U_n^{-1}F_n\varphi^d(U_n), \quad G_{n+1} = G_n + p^n[\varphi^{-m}(\overline{X}_n)],$$

where the Teichmüller map is applied to matrices entry by entry. The product $U_1U_2\cdots$ converges p -adically to a matrix U for which $U^{-1}F\varphi^d(U)$ is equal to the p -adic limit of the G_n , which is invertible over $W^1(L)$. Thus the vectors $\mathbf{e}'_j = \sum_i U_{ij}\mathbf{e}_i$ form a basis of the desired form. \square

Theorem 2.4.5. *Base extension of étale (φ, Γ) -modules from $\tilde{\mathbf{A}}_K^\dagger$ to $\tilde{\mathbf{A}}_K$ is an equivalence of categories. Consequently (by Theorem 2.3.5), both categories are equivalent to the category of continuous representations of G_K on finite free \mathbb{Z}_p -modules.*

Proof. We first check that the base extension functor is fully faithful. Again by Remark 2.4.1, this reduces to checking that if M is an étale (φ, Γ) -module over $\tilde{\mathbf{A}}_K^\dagger$, then

$$M^{\varphi, \Gamma} = (M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K)^{\varphi, \Gamma}.$$

By Theorem 2.3.5, $M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K$ corresponds to a continuous representation of G_K on a finite free \mathbb{Z}_p -module T . We can then find a finite extension L of K such that G_L acts trivially on T/pT ; this means that $M/pM \otimes_{\tilde{\mathbf{A}}_K^\dagger/(p)} \tilde{\mathbf{A}}_L^\dagger/(p)$ admits a basis fixed by both φ and Γ . By Lemma 2.4.2,

$$(M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K)^{\varphi, \Gamma} \subseteq (M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_L)^{\varphi, \Gamma} = (M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_L^\dagger)^{\varphi, \Gamma}.$$

However, within $\tilde{\mathbf{A}}_L$ we have

$$\tilde{\mathbf{A}}_L^\dagger \cap \tilde{\mathbf{A}}_K = \tilde{\mathbf{A}}_K^\dagger$$

and likewise after tensoring with the finite free module M . Hence $(M \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K)^{\varphi, \Gamma} \subseteq M^{\varphi, \Gamma}$ as desired.

It remains to check that the base extension functor is essentially surjective. For a given étale (φ, Γ) -module M over $\tilde{\mathbf{A}}_K$, to show that M descends to $\tilde{\mathbf{A}}_K^\dagger$, it is enough to check that it descends to $\tilde{\mathbf{A}}_L^\dagger$ for some finite Galois extension L of K (as then Lemma 2.4.2 provides the data needed to perform Galois descent back to K). Consequently, using Theorem 2.3.5, we may again reduce to the case where M admits a basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ which is fixed by φ and Γ modulo p .

By Lemma 2.4.4 applied to each component of $\tilde{\mathbf{A}}_K/(p)$, we obtain an étale φ -module M^\dagger over $\tilde{\mathbf{A}}_K^\dagger$ and an isomorphism $M^\dagger \otimes_{\tilde{\mathbf{A}}_K^\dagger} \tilde{\mathbf{A}}_K \cong M$ of φ -modules. By Lemma 2.4.2 and Lemma 2.4.3, the action of Γ on M induces an action on M^\dagger which is continuous for the weak and LF topologies. This yields the desired result. \square

Exercise 2.4.6. One can omit the requirement of continuity of the action of Γ in the definition of an étale (φ, Γ) -module, as it is implied by the other properties. (Hint: first do the case over $\tilde{\mathbf{A}}_K$, which implies the case over \mathbf{A}_K . Then use Theorem 2.4.5 to deduce the case over $\tilde{\mathbf{A}}_K^\dagger$, which implies the case over \mathbf{A}_K^\dagger .)

2.5. More on the action of Γ . We have seen that descent of étale (φ, Γ) -modules from $\tilde{\mathbf{A}}_K$ to $\tilde{\mathbf{A}}_K^\dagger$ can be achieved mainly using the bijectivity of φ . To make a similar passage from \mathbf{A}_K to \mathbf{A}_K^\dagger , we trade the failure of this bijectivity for better control of the action of Γ .

Lemma 2.5.1. *There exists $c > 0$ such that for $\bar{x} \in \mathbf{A}_K/(p)$, for n a positive integer, for $\gamma \in 1 + p^n \mathbb{Z}_p \subseteq \Gamma$,*

$$|(\gamma - 1)(\bar{x})'| \leq cp^{-p^{n+1}/(p-1)} |\bar{x}'|.$$

Proof. For $K = \mathbb{Q}_p$, for $\bar{x} = \bar{\pi}$,

$$(\gamma - 1)(\bar{\pi}) = (\gamma - 1)(1 + \bar{\pi}) = (1 + \bar{\pi})((1 + \bar{\pi})^{\gamma-1} - 1)$$

is divisible by $\bar{\pi}^{p^n}$, so $|(\gamma - 1)(\bar{\pi})'| \leq (|\bar{\pi}'|)^{p^n} = p^{-p^{n+1}/(p-1)} |\bar{\pi}'|$. This implies the general result for $K = \mathbb{Q}_p$ with $c = 1$.

For general K , it is similarly sufficient to check the claim for $\bar{x} = \bar{\pi}_L$ a uniformizer of a connected component L of $\mathbf{A}_K/(p)$. Let $P(T) = \sum_i P_i T^i$ be the minimal polynomial of $\bar{\pi}_L$ over $\mathbb{F}_p((\bar{\pi}))$, and put $Q(T) = P(T + \bar{\pi}_L) = \sum_{i>0} Q_i T^i$ with $Q_1 \neq 0$. Then

$$0 = \gamma(P(\bar{\pi}_L)) = (\gamma - 1)(P)(\gamma(\bar{\pi}_L)) + P(\gamma(\bar{\pi}_L)) = \sum_i (\gamma - 1)(P_i) \gamma(\bar{\pi}_L)^i + Q(\gamma(\bar{\pi}_L) - \bar{\pi}_L).$$

Since γ acts continuously on $\mathbf{A}_K/(p)$, $|(\gamma - 1)(\bar{\pi}_L)'| \rightarrow 0$ as $n \rightarrow \infty$; hence for large n , $|Q(\gamma(\bar{\pi}_L) - \bar{\pi}_L)'| = |Q_1'| |(\gamma - 1)(\bar{\pi}_L)'|$. It follows that for large n ,

$$|(\gamma - 1)(\bar{\pi}_L)'| = |Q_1^{-1}'| \left| \sum_i (\gamma - 1)(P_i) \gamma(x)^i \right|' \leq p^{-p^{n+1}/(p-1)} |Q_1^{-1}'| \max_i \{|P_i'| |\bar{\pi}_L^i|\}';$$

this implies the desired result. \square

Remark 2.5.2. Note that Lemma 2.5.1 is a far cry from what happens over $\tilde{\mathbf{A}}_K/(p)$: one cannot establish an inequality of the form $|(\gamma - 1)(x)'| \leq c|x'|$ uniformly over $x \in \tilde{\mathbf{A}}_K/(p)$ for even a single choice of $\gamma \in \Gamma - \{1\}$ and $c < 1$. (If one had such an inequality, then substituting $x^{1/p}$ in place of x would immediately imply the same inequality with c replaced by c^p , which would ultimately force $|(\gamma - 1)(x)'| = 0$ for all x .)

Lemma 2.5.3. *There exists $c > 0$ (depending on K) such that for any positive integer m and any $\bar{x} = \sum_{e=0}^{p^m-1} (1 + \pi)^{e/p^m} \bar{x}_{e/p^m}$ with $\bar{x}_{e/p^m} \in \mathbf{A}_K/(p)$,*

$$\max_e \{|\bar{x}_{e/p^m}'|\} \leq c|\bar{x}'|.$$

Proof. We first produce $c_0 \geq 1$ that satisfies the claim for $m = 1$. To do this, note that $|\bar{x}'|$ and $\max_e \{|\bar{x}_{e/p}'|\}$ both define norms on the finite-dimensional vector space $\varphi^{-1}(\mathbf{A}_K/(p))$ over $\mathbb{F}_p((\bar{\pi}))$. The claim then follows from the fact that any two norms on a finite-dimensional vector space over a complete field are equivalent (e.g., see [9, Theorem 1.3.6]).

We next show by induction on m that for any nonnegative integer m and any $\bar{x} = \sum_{e=0}^{p^m-1} (1 + \pi)^{e/p^m} \bar{x}_{e/p^m}$ with $\bar{x}_{e/p^m} \in \mathbf{A}_K/(p)$,

$$\max_e \{|\bar{x}_{e/p^m}'|\} \leq c_0^{1+1/p+\dots+1/p^{m-1}} |\bar{x}'|.$$

This is vacuously for $m = 0$. For $m > 0$, if the claim is known for $m - 1$, then by the previous paragraph we can write $\bar{x}^{p^{m-1}} = \sum_{f=0}^{p-1} (1 + \pi)^{f/p} \bar{y}_{f/p}$ with $\bar{y}_{f/p} \in \mathbf{A}_K/(p)$

and $\max_f \{|\bar{y}_{f/p}'|\} \leq c_0 |\bar{x}^{p^{m-1}}|'$. By the induction hypothesis, we can then write $\bar{y}_{f/p}^{p^{-m+1}} = \sum_{g=0}^{p^{m-1}-1} (1+\pi)^{g/p^{m-1}} \bar{z}_{f/p,g/p^{m-1}}$ with $\bar{z}_{f/p,g/p^{m-1}} \in \mathbf{A}_K/(p)$ and

$$\max_g \{|\bar{z}_{f/p,g/p^{m-1}}|\} \leq c_0^{1+1/p+\dots+1/p^{m-2}} |\bar{y}_{f/p}^{p^{-m+1}}|'.$$

For $e = 0, \dots, p^m - 1$, write $e = p^{m-1}f + g$ with $f \in \{0, \dots, p-1\}$ and $g \in \{0, \dots, p^{m-1}-1\}$ and set $\bar{x}_{e/p^m} = \bar{z}_{f/p,g/p^{m-1}}$; then $\bar{x} = \sum_{e=0}^{p^m-1} (1+\pi)^{e/p^m} \bar{x}_{e/p^m}$ and

$$\max_e \{|\bar{x}_{e/p^m}'|\} \leq c_0^{1+1/p+\dots+1/p^{m-2}} \max_f \{|\bar{y}_{f/p}^{p^{-m+1}}|\} \leq c_0^{1+1/p+\dots+1/p^{m-1}} |\bar{x}'|.$$

This completes the induction; we may now take

$$c = c_0^{1+1/p+1/p^2+\dots} = c_0^{p/(p-1)}$$

and deduce the desired result. \square

Corollary 2.5.4. *Let $\bar{T} \subset \tilde{\mathbf{A}}_K/(p)$ be the closure of the subgroup generated by $(1+\bar{\pi})^e \mathbf{A}_K/(p)$ for all $e \in \mathbb{Z}[p^{-1}] \cap (0, 1)$. Then the natural map $\mathbf{A}_K/(p) \oplus \bar{T} \rightarrow \tilde{\mathbf{A}}_K/(p)$ is an isomorphism of Banach spaces over $\mathbb{F}_p((\bar{\pi}))$.*

Exercise 2.5.5. Compute the optimal constant c in Lemma 2.5.3 for $K = \mathbb{Q}_p$.

Lemma 2.5.6. *For \bar{T} as in Corollary 2.5.4, for any $\gamma \in \Gamma - \{1\}$, the map $\gamma - 1 : \bar{T} \rightarrow \bar{T}$ is bijective with bounded inverse.*

Proof. By Lemma 2.5.3, it is sufficient to check that for each $e \in \mathbb{Z}[p^{-1}] \cap (0, 1)$, $\gamma - 1$ is bijective on $(1+\bar{\pi})^e \mathbf{A}_K/(p)$ with the inverse bounded uniformly in e . Using the identity

$$(\gamma - 1)^{-1} = (1 + \gamma + \dots + \gamma^{m-1})(\gamma^m - 1)^{-1},$$

we may reduce the claim for γ to the claim for γ^m for any convenient positive integer m (chosen uniformly in e). Consequently, we may assume that $\gamma \in (1 + p^n \mathbb{Z}_p) - (1 + p^{n+1} \mathbb{Z}_p)$ for n chosen large enough that there exists c as in Lemma 2.5.1 less than p^n .

For $\bar{x} \in \mathbf{A}_K/(p)$ we may write

$$\begin{aligned} (\gamma - 1)((1 + \bar{\pi})^e \bar{x}) &= (\gamma - 1)((1 + \bar{\pi})^e) \bar{x} + \gamma((1 + \bar{\pi})^e)(\gamma - 1)(\bar{x}) \\ &= (1 + \bar{\pi})^e ((1 + \bar{\pi})^{(\gamma-1)e} - 1) \bar{x} + (1 + \bar{\pi})^{\gamma e} (\gamma - 1)(\bar{x}). \end{aligned}$$

By Lemma 2.5.1, $|(\gamma - 1)(\bar{x})'| \leq cp^{-p^{n+1}/(p-1)} |\bar{x}'|$. On the other hand, since $e \in \mathbb{Z}[p^{-1}] \cap (0, 1)$, $(\gamma - 1)e$ has p -adic valuation at most $n - 1$, so $|(1 + \bar{\pi})^{(\gamma-1)e} - 1|' \geq (|\bar{\pi}'|)^{p^{n-1}} = p^{-p^n/(p-1)}$. Since $cp^{-p^{n+1}/(p-1)} < p^{-p^n/(p-1)}$ by our choice of n , we have

$$|(\gamma - 1)((1 + \bar{\pi})^e \bar{x})'| = |((1 + \bar{\pi})^{(\gamma-1)e} - 1) \bar{x}'| \geq p^{-p^n/(p-1)} |\bar{x}'|.$$

This proves the claim. \square

Corollary 2.5.7. *For \bar{T} as in Corollary 2.5.4 and $\gamma \in \Gamma - \{1\}$, every $\bar{x} \in \tilde{\mathbf{A}}_K/(p)$ can be written uniquely as $\bar{y} + (\gamma - 1)(\bar{z})$ with $\bar{y} \in \mathbf{A}_K/(p)$, $\bar{z} \in \bar{T}$. Moreover,*

$$\max\{|\bar{y}'|, |\bar{z}'|\} \leq c|\bar{x}'|$$

for some constant c (depending on K and γ but not on \bar{x}).

Corollary 2.5.8. *Let $T \subset \tilde{\mathbf{A}}_K$ be the closure for the weak topology of the subgroup generated by $(1 + \pi)^e \mathbf{A}_K$ for all $e \in \mathbb{Z}[p^{-1}] \cap (0, 1)$. For $\gamma \in \Gamma - \{1\}$, every $x \in \tilde{\mathbf{A}}_K$ can be written uniquely as $y + (\gamma - 1)(z)$ with $y \in \mathbf{A}_K$, $z \in T$. Moreover, there exist $c, r_0 > 0$ (depending on K and γ) such that*

$$\max\{|y|_r, |z|_r\} \leq c^r |x|_r \quad (r \in (0, r_0]).$$

Proof. Combine Corollary 2.5.7 with Corollary 2.2.10. □

2.6. Overconvergence and (φ, Γ) -modules, part 2: the theorem of Cherbonnier-Colmez. We start with the following analogue of Lemma 2.4.2.

Lemma 2.6.1. *Base extension of étale φ -modules which are trivial modulo p from \mathbf{A}_K^\dagger to \mathbf{A}_K is fully faithful.*

Proof. Again by Remark 2.4.1, this reduces to checking that if M is an étale φ -module over \mathbf{A}_K^\dagger which is trivial modulo p , then

$$M^\varphi = (M \otimes_{\mathbf{A}_K^\dagger} \mathbf{A}_K)^\varphi.$$

By Lemma 2.4.2, we already have

$$(M \otimes_{\mathbf{A}_K^\dagger} \mathbf{A}_K)^\varphi \subseteq (M \otimes_{\mathbf{A}_K^\dagger} \tilde{\mathbf{A}}_K)^\varphi = (M \otimes_{\mathbf{A}_K^\dagger} \tilde{\mathbf{A}}_K^\dagger)^\varphi.$$

Since M is a free module, within $M \otimes_{\mathbf{A}_K^\dagger} \tilde{\mathbf{A}}_K$ we have

$$(M \otimes_{\mathbf{A}_K^\dagger} \mathbf{A}_K) \cap (M \otimes_{\mathbf{A}_K^\dagger} \tilde{\mathbf{A}}_K^\dagger) = M,$$

yielding the desired result. □

Theorem 2.6.2 (Cherbonnier-Colmez). *Base extension of étale (φ, Γ) -modules from \mathbf{A}_K^\dagger to \mathbf{A}_K is an equivalence of categories. Consequently (by Theorem 2.3.5), both categories are equivalent to the category of continuous representations of G_K on finite free \mathbb{Z}_p -modules.*

Proof. By Theorem 2.3.5 and Theorem 2.4.5, it is equivalent to show that base extension of étale (φ, Γ) -modules from \mathbf{A}_K^\dagger to $\tilde{\mathbf{A}}_K^\dagger$ is an equivalence of categories. As in the proof of Theorem 2.4.5 (but now using Lemma 2.6.1 instead of Lemma 2.4.2), it is sufficient to check that any étale (φ, Γ) -module M over $\tilde{\mathbf{A}}_K^\dagger$ admitting a basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ fixed by φ and Γ modulo p descends to \mathbf{A}_K^\dagger .

Put $\gamma = 1 + p^2 \in \Gamma$. Define T, c, r_0 as in Corollary 2.5.8. Let $G \in \mathrm{GL}_d(\tilde{\mathbf{A}}_K^\dagger)$ be given by $\gamma(\mathbf{e}_j) = \sum_i G_{ij} \mathbf{e}_i$, so that $G - 1$ is divisible by p . We can then choose $r \in (0, r_0]$ so that $\epsilon = |G - 1|_r^{1/3} < \min\{c^{-r}, 1\}$.

We define a sequence of invertible matrices U_0, U_1, \dots over $\tilde{\mathbf{A}}_K^\dagger$ congruent to 1 modulo p with the property that $G_l = U_l^{-1} G \gamma(U_l)$ can be written as $1 + X_l + (\gamma - 1)(Y_l)$ with X_l having entries in \mathbf{A}_K^\dagger , Y_l having entries in T , and

$$|X_l|_r \leq \epsilon^2, \quad |Y_l|_r \leq \epsilon^{l+2}.$$

To begin with, put $U_0 = 1$ and apply Corollary 2.5.8 to construct X_0, Y_0 of the desired form with $|X_0|_r, |Y_0|_r \leq c^r |G - 1|_r \leq \epsilon^2$. Given U_l , set $U_{l+1} = U_l(1 - Y_l)$ and write

$$\begin{aligned} G_{l+1} &= (1 - Y_l)^{-1} (1 + X_l + (\gamma - 1)(Y_l)) (1 - \gamma(Y_l)) \\ &= 1 + X_l + Y_l X_l - X_l \gamma(Y_l) + E_l \end{aligned}$$

with $|E_l|_r \leq \epsilon^{2l+4}$. Note that $Y_l X_l - X_l \gamma(Y_l)$ has entries in T and $|Y_l X_l - X_l \gamma(Y_l)|_r \leq \epsilon^{l+4}$. Apply Corollary 2.5.8 to split $Y_l X_l - X_l \gamma(Y_l) + E_l$ as $A_l + (\gamma - 1)(B_l)$ with A_l having entries in \mathbf{A}_K^\dagger , B_l having entries in T , and $|A_l|_r, |B_l|_r \leq c^r \epsilon^{l+4} \leq \epsilon^{l+3}$. Set $X_{l+1} = X_l + A_l$, $Y_{l+1} = B_l$ and continue.

The product $U = U_0 U_1 \cdots$ converges to an invertible matrix U over $\tilde{\mathbf{A}}_K^\dagger$. Define the basis $\mathbf{e}'_1, \dots, \mathbf{e}'_d$ of M by $\mathbf{e}'_j = \sum_i U_{ij} \mathbf{e}_i$. Define the matrices A, H by $\varphi(\mathbf{e}'_j) = \sum_i A_{ij} \mathbf{e}_i$, $\gamma(\mathbf{e}'_j) = \sum_i H_{ij} \mathbf{e}_i$. By construction, H has entries in \mathbf{A}_K^\dagger and is congruent to 1 modulo p . Since φ and γ commute, $A\varphi(H) = H\gamma(A)$. Apply Corollary 2.5.8 to write $A = B + C$ with B having entries in \mathbf{A}_K^\dagger and C having entries in T ; then

$$H^{-1}C\varphi(H) - C = (\gamma - 1)(C).$$

If C is nonzero, then there is a largest nonnegative integer m such that C is divisible by p^m . However, since $H \equiv 1 \pmod{p}$, $H^{-1}C\varphi(H) - C$ is divisible by p^{m+1} while $(\gamma - 1)(C)$ is not, a contradiction. Hence $C = 0$ and $A = B$ has entries in \mathbf{A}_K^\dagger .

Let M^\dagger be the \mathbf{A}_K^\dagger -span of $\mathbf{e}'_1, \dots, \mathbf{e}'_d$; it is an étale φ -module over \mathbf{A}_K^\dagger such that $M^\dagger \otimes_{\mathbf{A}_K^\dagger} \tilde{\mathbf{A}}_K^\dagger \cong M$. By Lemma 2.6.1, the action of Γ descends to M^\dagger ; it is automatically continuous because \mathbf{A}_K^\dagger and $\tilde{\mathbf{A}}_K^\dagger$ carry the same topologies. This proves the desired result. \square

REFERENCES

- [1] L. Berger, An introduction to the theory of p -adic representations, in *Geometric Aspects of Dwork Theory*, volume I, Walter de Gruyter, Berlin, 2004, 255–292.
- [2] L. Berger and P. Colmez, Familles de représentations de de Rham et monodromie p -adique, *Astérisque* **319** (2008), 303–337.
- [3] O. Brinon and B. Conrad, Lecture notes on p -adic Hodge theory, available at <http://math.stanford.edu/~conrad/>.
- [4] F. Cherbonnier and P. Colmez, Représentations p -adiques surconvergentes, *Invent. Math.* **133** (1998), 581–611.
- [5] L. Fargues and J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge p -adique, in preparation; draft (December 2010) available at <http://www.math.u-psud.fr/~fargues/Prepublications.html>.
- [6] J.-M. Fontaine, Représentations p -adiques des corps locaux, I, in *The Grothendieck Festschrift, Vol. II*, Progr. Math. 87, Birkhäuser, Boston, 1990, 249–309.
- [7] J.-M. Fontaine and J.-P. Wintenberger, Le “corps des normes” de certaines extensions algébriques de corps locaux, *C.R. Acad. Sci. Paris Sér. A-B* **288** (1979), A367–A370.
- [8] O. Gabber and L. Ramero, *Almost Ring Theory*, Lecture Notes in Math. 1800, Springer-Verlag, Berlin, 2003.
- [9] K.S. Kedlaya, *p -adic Differential Equations*, Cambridge Studies in Advanced Math. 125, Cambridge Univ. Press, Cambridge, 2010.
- [10] K.S. Kedlaya, Nonarchimedean geometry of Witt vectors, arXiv:1004.0466v4 (2011).
- [11] K.S. Kedlaya and R. Liu, Relative p -adic Hodge theory, I: Foundations, preprint (2011) available at <http://math.mit.edu/~kedlaya/papers/>.
- [12] K.S. Kedlaya and R. Liu, Relative p -adic Hodge theory, II: (φ, Γ) -modules, preprint (2011) available at <http://math.mit.edu/~kedlaya/papers/>.
- [13] P. Scholze, Perfectoid spaces, I: Basic theory, and the weight-monodromy conjecture, in preparation.
- [14] P. Scholze, Perfectoid spaces, II: p -adic Hodge theory, in preparation.
- [15] S. Sen, Ramification in p -adic Lie extensions, *Inv. Math.* **17** (1972), 44–50.
- [16] J.-P. Serre, *Local Fields*, Graduate Texts in Math. 67, Springer-Verlag, New York, 1979.