REPRESENTATIONS OF QUANTUM GROUPS AT
ROOTS OF 1: REDUCTION TO THE EXCEPTIONAL CASE

CORRADO DE ConcINI
Scuola Normale Superiore
Pisa, Italy

and

VICTOR G. KAC
Department of Mathematics, MIT
Cambridge, MA 02139, USA

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ABSTRACT

This paper is a continuation of the papers [DC–K] and [DC–K–P] on representations of quantum groups at roots of 1. Here we show that an irreducible representation of a quantum group at an odd root of 1 can be uniquely induced from an exceptional representation of a smaller quantum group. This reduces the classification of representations, the calculation of their characters and dimensions, etc, to the exceptional case.

§1. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, let $\mathfrak{h}$ be its Cartan subalgebra, let $R \subset \mathfrak{h}^*$ be the set of roots, let $Q = \mathbb{Z}R$ be the root lattice, and let $W \subset \text{Aut} \, \mathfrak{h}^*$ be the Weyl group. Choose a subset of positive roots $R^+ \subset R$, let $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset R^+$ be the set of simple roots and let $s_1, \ldots, s_n$ be the corresponding simple reflections generating $W$. Let $(\cdot, \cdot)$ be a $W$-invariant bilinear form on $\mathfrak{h}^*$ normalized by the condition that the square length of a short root equals 2. Then

$$(\alpha_i | \alpha_j) = d_i a_{ij}, \quad i, j = 1, \ldots, n,$$

where $d_1, \ldots, d_n$ are relatively prime positive integers and $(a_{ij})$ is the Cartan matrix of $\mathfrak{g}$.

Recall that connected Lie groups with Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with lattices $M$ containing $Q$ such that $(\lambda | d_j^{-1} \alpha_j) \in \mathbb{Z}$ for all $j = 1, \ldots, n$. Fix such a lattice $M$ and let $G$ be the corresponding connected Lie group (so that Center $G = M/Q$).
§2. Recall that the "quantum group at $\varepsilon$" is the associative algebra $U = U_M, (g)$ over $\mathbb{C}$ on generators $E_i, F_i (i = 1, \ldots, n), K_\alpha (\alpha \in M)$ and the following defining relations $(\alpha, \beta \in M, i, j = 1, \ldots, n)$:

(2.1) $K_\alpha K_\beta = K_{\alpha + \beta}, \ K_0 = 1$.
(2.2) $K_\alpha E_i K_{-\alpha} = \varepsilon^{(\alpha, \alpha)} E_i, \ K_\alpha F_i K_{-\alpha} = \varepsilon^{(-\alpha, \alpha)} F_i$.
(2.3) $E_i F_j - F_j E_i = \delta_{ij}(K_\alpha - K_{-\alpha})/(\varepsilon^{d_i} - \varepsilon^{-d_i})$.
(2.4) Certain Chevalley-Serre type relations between the $E_i$ and between the $F_i$ (see e.g. [L] or [DC-K, (1.2.4 and 5)]).

Let $\omega$ be a conjugate-linear anti-automorphism of $U$ defined by: $\omega E_i = F_i, \omega F_i = E_i, \omega K_\alpha = K_{-\alpha}$.

Let $U^+, U^-$ and $U^0$ be the subalgebras of $U$ generated by the $E_i$, by the $F_i (i = 1, \ldots, n)$ and by the $K_\alpha (\alpha \in M)$ respectively. Then multiplication defines a $\mathbb{C}$-vector space isomorphism $[R]

U = U^- \otimes U^0 \otimes U^+.$

§3. Recall that the braid group $B_W$ (associated to $W$) acts by automorphisms of $U$ defined by $[L]$ $(i = 1, \ldots, n)$:

$T_i K_\alpha = K_{s_i(\alpha)},$

$T_i E_i = -F_i K_i, \ T_i F_i = \sum_{s=0}^{d_i} (-1)^{s-a_{ij}} \varepsilon_i^{s} E_i^{(-a_{ij}-1)} E_i^{(s)}$ if $i \neq j,$

$T_i F_i = -K_i^{-1} E_i, \ T_i E_i = \sum_{s=0}^{d_i} (-1)^{s-a_{ij}} \varepsilon_i^{s} F_i^{(-a_{ij}-1)} F_i^{(s)}$ if $i \neq j.$

Here and further $E_i^{(a)}$ and $F_i^{(a)}$ stand for $E_i^a / [a]_d!$ and $F_i^a / [a]_d!$, where $[a]_d! = [a][a-1] \cdots [1]$ and $[a]_d = (\varepsilon^a - \varepsilon^{-a})/(\varepsilon^d - \varepsilon^{-d})$. Note that $T_i \omega = \omega T_i$.

Choosing a reduced expression $s_i, s_{i_2}, \ldots, s_{i_N}$ of the longest element of $W(N = |R^+|)$, we get a total ordering of $R^+$:

$\beta_1 = \alpha_{i_1}, \beta_2 = \alpha_{i_2}, \ldots, \beta_N = \alpha_{i_1} \cdots i_{N-1} \alpha_{i_N},$

and the corresponding root vectors ($k = 1, \ldots, N$):

$E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}} E_{i_k}, \ F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}} F_{i_k} = \omega E_{\beta_k}$

(they depend on the choice of the reduced expression).

For $k = (k_1, \ldots, k_N) \in \mathbb{Z}_+^N$ we let $E^k = E_{\beta_1} \cdots E_{\beta_N}$, $F^k = \omega E^k$.

Lemma 3.1. [L] (a) Elements $E^k$ (resp $F^k$), $k \in \mathbb{Z}_+^N$, form a basis of $U^+$ (resp $U^-$) over $\mathbb{C}$.

(b) Elements $F^k K_{\alpha} E^r$, where $k, r \in \mathbb{Z}_+^N$, $\alpha \in M$, form a basis of $U$ over $\mathbb{C}$.
Lemma 3.2. [L-S] For \( i < j \) one has:

\[
E_{\beta_j} E_{\beta_i} - e^{(\beta_i | \beta_j)} E_{\beta_j} E_{\beta_i} = \sum_{k \in H^*_N} c_k E^k,
\]

where \( c_k \in \mathbb{C} \) and \( c_k \neq 0 \) only when \( k = (k_1, \ldots, k_N) \) is such that \( k_s = 0 \) for \( s \leq i \) and \( s \geq j \).

\( \square \)

§4. Let \( Z \) denote the center of the algebra \( U \).

Lemma 4.1. [DC-K] Elements \( E^\alpha, F^\alpha, K^\beta \) (\( \alpha \in R^+, \beta \in M \)) lie in \( Z \).

Let \( Z_0 \) (resp. \( Z_0^- \) or \( Z_0^+ \)) be the subalgebra of \( Z \) generated by all the elements \( E^\alpha, F^\alpha, K^\beta \) (resp. \( K^\beta \) or \( E^\alpha \) or \( F^\alpha \)). By (2.5) we have:

\[
Z_0 = Z_0^- \otimes Z_0^+ \otimes Z_0^1.
\]

Now Lemma 3.1 implies

Lemma 4.2. [DC-K] The algebra \( U \) is a free \( Z_0 \)-module on the basis \( \{F^k K^\alpha E^r \} \), where \( k = (k_1, \ldots, k_N) \) and \( r = (r_1, \ldots, r_N) \) are such that \( 0 \leq k_i < l, 0 \leq r_i < l \) and \( \alpha \) runs over a basis of \( M \) mod \( IM \).

Given a homomorphism \( \chi : Z_0 \to \mathbb{C} \), let

\[
U_\chi = U/(z - \chi(z), \text{ where } z \in Z_0).
\]

Corollary 4.1. \( U_\chi \) is an algebra of dimension \( \dim \tilde{u} \) with a basis over \( \mathbb{C} \) described by Lemma 4.2.

\( \square \)

§5. Let \( A \) be the algebra of rational functions in \( q \) that have no poles at \( \varepsilon \).

Let \( U_A \) be the algebra over \( A \) on generators \( E_i, F_i, \text{ and } K^\alpha \) and defining relations (2.1)-(2.4) where \( \varepsilon \) is replaced by \( q \), so that \( U = U_A/(q - \varepsilon) \). Suppose that we have an element \( b \in U_A \) with the property that \( [b, a] \in (q - \varepsilon) U_A \) for all \( a \in U_A \). Then of course the image of \( b \) in \( U \) is central. Moreover one can also define a derivation \( P_b \) of \( U \) by

\[
P_b(a) = (q - \varepsilon)^{-1} [b, a] \mod (q - \varepsilon),
\]

where \( \tilde{a} \) is a preimage of \( a \) in \( U_A \). In particular, we have derivations \( e_i \) and \( f_i \) of \( U \) given by [DC-K] (in a slightly different normalization):

\[
e_i = P_{E^i}, \quad f_i = P_{F^i}.
\]

It was shown in [DC-K] that the series \( \exp t e_i \) and \( \exp t f_i, (t \in \mathbb{C}) \) converge to analytic automorphisms of certain analytic completion \( \tilde{U} \) of the algebra \( U \). Denote by \( \tilde{G} \) the group of automorphisms of \( \tilde{U} \) generated by all these 1-parameter groups.
The group $\tilde{G}$ leaves the completion of $Z_0$ invariant [DC-K]. Hence it acts on $\text{Spec } Z_0$ by $(\tilde{g}x)(z) = x(\tilde{g}^{-1}(z)), \tilde{g} \in \tilde{G}$, and we have an isomorphism of algebras:

\[(5.1) \quad \tilde{g} : U_x \cong U_{\tilde{g}(x)}, \quad \tilde{g} \in \tilde{G}.
\]

This induces a canonical bijection (for the definition of $\text{Spec}$ see below)

\[(5.2) \quad \tilde{g} : \text{Spec } U_x \rightarrow \text{Spec } U_{\tilde{g}(x)},
\]

where $(\tilde{g}\sigma)(u) := \sigma(\tilde{g}^{-1}u), u \in U_x$.

§6. Let $G'$ be the connected cover of $G$ with fundamental group $\pi_1(G') = \pi_1(G)/\pi_1(G)^0$. Denote by $\text{Spec } A$ the set of all equivalence classes of irreducible finite-dimensional representations of an algebra $A$. Recall that we have the following sequence of canonical maps:

\[(6.1) \quad \text{Spec } U \rightarrow \text{Spec } Z \rightarrow \text{Spec } Z_0 \rightarrow G'.
\]

Here $X$ is the map of taking central characters, $\tau$ is the restriction map and $\pi$ is a map constructed in [DC-K-P]. The maps $X$ and $\tau$ are surjective, the map $\chi$ is bijective over a Zariski open dense subset of $\text{Spec } Z$ and has finite fibers, the map $\tau$ is finite with fibers of order $\leq p^n$, which are explicitly described ([DC-K],[DC-K-P]). Note also that a representation $\sigma \in \text{Spec } U$ with $\chi = X(\sigma)$ is actually a representation of the algebra $U_x$.

In order to describe properties of the map $\pi$ which will be needed in the sequel, introduce some notation. Let $T$ (resp. $T'$) be the maximal torus of $G$ (resp. $G'$) corresponding to $h \subset g$, and let $N_-$ and $N_+$ be maximal unipotent subgroups of $G'$ corresponding to $-R^+$ and $R^+$ respectively. We shall identify $\text{Spec } Z_0^0$ with $T$ via the isomorphism $M \cong IM$ given by multiplication by $I$. Recall that multiplication in $G'$ defines a birational isomorphism $N_- \times T' \times N_+ \rightarrow N_-T'N_+ = G^0$, where $G^0$ is a Zariski open dense subset of $G'$ (called the big cell of $G'$). Given a conjugacy class $\mathcal{O}$ of $G'$ we let $\mathcal{O}^0 = \mathcal{O} \cap G^0$; this is a Zariski open dense subset of $\mathcal{O}$.

**Lemma 6.1.** [DC-K-P] (a) We have:

\[\pi = \pi^- \times \pi^0 \times \pi^+ : \text{Spec } Z_0^- \times \text{Spec } Z_0^0 \times \text{Spec } Z_0^+ \rightarrow \text{Spec } Z_0 \rightarrow \text{Spec } Z_0 \rightarrow G',\]

where $\pi^\pm : \text{Spec } Z_0^\pm \rightarrow N_\pm$ is a birational isomorphism and $\pi^0 : T \rightarrow T'$ is a homomorphism given by the square map.

(b) The set $F$ of fixed points of $\tilde{G}$ in $\text{Spec } Z_0$ is $(\pi^0)^{-1}(\text{Center } G') \subset T = \text{Spec } Z_0^0$.

(c) If $\mathcal{O}$ is a conjugacy class of a non-central element of $G'$, then $\pi^{-1}(\mathcal{O}^0)$ is a single $G$-orbit and $(\text{Spec } Z_0)\setminus F$ is a union of these $G$-orbits.

(d) If $\chi_- \in \text{Spec } Z_0^-$ and $\chi_0 \in \text{Spec } Z_0^0$ are such that $\pi^-(\chi_-)$ and $\pi^0(\chi_0)$ are commuting elements of $G'$ and $\chi_0(K_{\tilde{G}}) \neq 1$ for some $\alpha \in R^+$, then $\chi_-(F_{\alpha}^i) = 0$. ◯

§7. We call a semisimple element $g$ of the algebraic group $G'$ exceptional if its centralizer in $G'$ has a finite center. All semisimple exceptional elements are classified by the following lemma which can be easily deduced from [K, Chapter 8]:

\[\text{Spec } U \rightarrow \text{Spec } Z \rightarrow \text{Spec } Z_0 \rightarrow G'.\]
Lemma 7.1. (a) Let \( \theta = \sum_{i=1}^{n} a_i \alpha_i \) be the highest root in \( \mathbb{R}^+ \). Define elements \( \omega_m^j \in \mathfrak{h} \) (\( m = 1, \cdots, n \)) by
\[
(\alpha_j, \omega_m^j) = \delta_{jm}, \quad j = 1, \cdots, n.
\]
Then elements \( s_m := \exp(2\pi i \omega_m^j / a_m) \in T' \subset G' \) and \( s_0 = 1 \) are exceptional semisimple elements and any exceptional semisimple element is conjugate to one of the \( s_m(m = 0, 1, \cdots, n) \).

(b) Up to multiplication by a central element the \( s_m \) give a complete non-redundant list of representatives of exceptional semisimple elements for the following \( m \) (the numbering of simple roots is taken from [K, Chapter 4]):

\[
\begin{array}{c|c}
A_n & m = 0 \\
B_n & 1 \leq m \leq n \\
C_n & 0 \leq m \leq \lfloor n/2 \rfloor \\
D_n & 0 \leq m \leq [(n - 1)/2] \\
E_6 & 3 \leq m \leq 6 \\
E_7 & 3 \leq m \leq 7 \\
E_8, F_4, G_2 & 0 \leq m \leq n \\
\end{array}
\]

\( \square \)

An element \( g \) of \( G' \) is called exceptional if its semisimple part is exceptional. In other words a complete set of representatives of conjugacy classes of exceptional elements is given by \( \{s_mu\} \), where \( u \) are representatives of conjugacy classes of unipotent elements in the centralizer of the \( s_m \). Note that the number of conjugacy classes of exceptional elements in \( G' \) is finite.

\section{Exceptions}

Let \( \varphi = \pi \circ \tau \circ X : \text{Spec} \, U \to G' \) be the composition of maps of the sequence (6.1). A finite-dimensional irreducible representation of \( U \) is called exceptional if its image in \( G' \) under the map \( \varphi \) is an exceptional element.

Suppose now that \( \sigma \) is a non-exceptional finite-dimensional irreducible representation of the algebra \( U \) in a vector space \( V \), and let \( \chi = X(\sigma) \in \text{Spec} \, Z \) so that \( \sigma \in \text{Spec} \, U_X \). Since the element \( \varphi(\sigma) \) is not exceptional, its conjugacy class in \( G' \) contains an element \( g \) with the following properties:

\[
(8.1) \quad g_s \in T', g_u \in N_-,
\]

where \( g_s \) and \( g_u \) denote the semisimple and unipotent parts of \( g \);

\[
(8.2) \quad \mathfrak{h}_g := \text{Lie}(\text{center of Centralizer } G'(g_s)) \neq 0;
\]

\[
(8.3) \quad R' := \{ \alpha \in R \mid \alpha \text{ vanishes on } \mathfrak{h}_g \} = M' \cap R,
\]

where \( M' = \mathbb{Z} \Pi' \) is a sublattice of \( M \) spanned by \( \mathbb{Z} \Pi \) of \( \Pi \) different from \( \Pi \).

By Lemma 6.1c, there exists an element \( \tilde{g} \in \tilde{G} \) such that \( \varphi(\tilde{g}(\sigma)) = g \). Replacing \( \sigma \) by \( \tilde{g}(\sigma) \) and \( \chi \) by \( \tilde{g}(\chi) \), we may assume that \( \sigma \) is an irreducible representation of the algebra \( U_X \) in the vector space \( V \), such that \( g := \varphi(\sigma) \) satisfies (8.1)–(8.3).

Let \( U' \) be the subalgebra of \( U \) generated by \( U_0^0 \) and all the elements \( E_i \) and \( F_i \) such that \( \alpha_i \in \Pi' \), and let \( U'_2 = U'/z - \chi(z) \), where \( z \in Z_0 \cap U' \). Let \( U'^3 = U'^{+} \cup U'^{+} / (z - \chi(z) \), where \( z \in Z_0 \cap U' \) be the corresponding "parabolic" subalgebras.

Now we are in a position to state the main theorem (Theorem 2 from [W-K] may be viewed as an "infinitesimal" analogue of this theorem).
Theorem. (a) The $U_X$-module $V$ contains a unique irreducible $U_X^{G}$-submodule $V'$, which is in fact a $U_X^{G}$-module.
(b) The $U_X$-module $V$ is induced from the $U_X^{G}$-module $V'$, i.e.
$$V = U_X \otimes_{U_X^G} V',$$
with the action of $U_X$ on $V$ defined by left multiplication on $U_X$. In particular,
$$\dim V = l \cdot \dim V', \quad \text{where} \quad 2l = |R \setminus R'|.$$
(c) The map $V \mapsto V'$ thus obtained establishes a bijection: Spec $U_X \mapsto$ Spec $U_X^G$.

Remark 8.1. The representation of $U_X^G$ in $V'$ remains irreducible when restricted to the subalgebra $U_X^a$ of $U_X^G$ generated by the $E_i$ and $F_i$ such that $\alpha_i \in \Pi'$ and by the $K_\beta$ such that $\beta \in M'$. This representation of $U_X^a$ is in fact an exceptional representation of the quantum group $U_{M',a}(g')$, where $g'$ is the subalgebra of $g$ generated by the Chevalley generators corresponding to $\alpha_i \in \Pi'$.

§9. The proof of this theorem is similar to that of Theorem 2 from [W-K] on irreducible representations of simple Lie algebras of characteristic $p$. It is based on several lemmas that we prove in this section.

Consider the root system $R'$. Let $R'^+$ be the corresponding subset of positive roots. Let $w_0^G$ be a reduced expression of the longest element of the Weyl group $W'$ of $R'$. We complete $w_0^G$ to a reduced expression of the longest element of $W$:

$$w_0 = w_0^G s_i \cdots s_i.$$  

Let
$$\gamma_i = \alpha_i \cdot \gamma_2 = s_i(\alpha_2), \ldots, \gamma_t = s_i \cdots s_{i-1}(\alpha_i).$$

Let $R^+_{(k)} = s_i \cdots s_i R^+ (k = 1, \ldots, t)$.

Lemma 9.1. (a) $R^+ \setminus R'^+ = \{ \gamma_1, \ldots, \gamma_t \}$.
(b) $\gamma_k$ is a simple root of $R^+_{(k)}$ and $\gamma_j \not\in -R^+_{(k)}$ for $j < k$.

Proof. It is clear that $\{ \gamma_1, \ldots, \gamma_t \} \subseteq R^+$ and that $w_0^G \{ \gamma_1, \ldots, \gamma_t \} \subseteq R^+$. This implies that $\{ \gamma_1, \ldots, \gamma_t \} \subseteq R^+ \setminus R'^+$. Since these two sets have equal cardinality, this proves (a).

It is clear by definition that $\gamma_k$ is simple in $R^+_{(k)}$. Since $(s_i \cdots s_{i-1})^{-1}s_i \cdots s_{i-1}\alpha_i = s_i \cdots s_i\alpha_i \in -R^+$, (b) follows. □

Note that we have the following important properties of the $\gamma_i$:

$$R^2_{\gamma_i} \neq 1, \quad i = 1, \ldots, t,$$

hence, by Lemma 6.1d,

$$R_{\gamma_i} = 0, \quad i = 1, \ldots, t.$$
Let $B$ be the subalgebra of $U_\chi$ generated by the $K_\alpha$ ($\alpha \in M$) and $E_i$ ($i = 1, \ldots, n$). Given $m \in \{1, 2, \ldots, n\}$, let $P_m$ denote the subalgebra of $U_\chi$ generated by $B$ and $F_m$. (In the sequel, we shall take $m = i_1$.) Taking a reduced expression of $w_0$ which starts with $s_m$, consider the corresponding root vectors $E_{\beta_1} = E_{m}, E_{\beta_2}, \ldots, E_{\beta_N}$. Denote by $N_m$ the subalgebra of $U_\chi$ generated by $E_{\beta_2}, \ldots, E_{\beta_N}$ and let $N_m$ be its 2-sided ideal generated by $E_{\beta_2}, \ldots, E_{\beta_N}$.

**Lemma 9.2**

(a) $F_m E_{\beta} - e^{(\alpha_1(\beta)} F_m E_{\beta} \in N_m$ for $\beta = \beta_2, \ldots, \beta_N$.

(b) $N_m$ is independent of the choice of the reduced expression (which starts with $s_m$).

**Proof.** (a) follows from formula (3.1) for $E_m$ and $E_{s_m(\beta)}$ by applying $T_m$ to both sides.

In order to prove (b) suppose for example that $w_0 = w r_1 r_2 w_1 = w r_2 r_1 w_1$. Then the corresponding root vectors are respectively:

$$
\{ \ldots, T_w E_1, T_w T_1 E_1, T_w T_1 T_1 E_1 = T_w E_2, \ldots \};

\{ T_w E_1, T_w T_2 E_1, T_w T_2 T_1 E_1 = T_w E_1, \ldots \}.
$$

Since $T_w(T_1 E_1)$ lies in the subalgebra generated by $T_w E_1$ and $T_w E_2$, this proves (b).

Let $B_{(1)} = B, N_{(1)} = N_m, \overline{N}_{(1)} = N_m, P_{(1)} = P_m, K_{(1)} = K_m, \text{ etc.}$

For a $B_{(1)}$-module $A$, we let

$$
A_{(1)} = \{ a \in A | \overline{N}_{(1)} a = 0 \}.
$$

**Lemma 9.3.** Let $A$ be a $B_{(1)}$-module. Let $V = P_{(1)} \otimes B_{(1)} A$ be the $P_{(1)}$-module induced by the $B_{(1)}$-module $A$. Then

(a) $V_{(1)}$ is $P_{(1)}$-stable.

(b) $V_{(1)}$ lies in $\sum_{k=0}^{l-1} F^k A_{(1)}$.

(c) If $E_{(1)} A_{(1)} \equiv 0$ and $K_{(1)}^l \neq 1$, then any $P_{(1)}$-submodule $C$ of $V_{(1)}$ intersects $A_{(1)}$ non-trivially.

**Proof.** (a) follows from Lemma 9.2a.

We shall write $E$ and $F$ in place of $E_{(1)}$ and $F_{(1)}$ to simplify notation. In order to prove (b), write $v \in V_{(1)}$ in the form:

$$
v = \sum_{k=0}^{s} F^k x_k, \quad \text{where} \quad s \leq l - 1, x_k \in A.
$$

If $\beta = \beta_2, \ldots, \beta_N$, we have:

$$
0 = E_{\beta} v = E_{\beta} F^s x_s + \sum_{k=0}^{s-1} E_{\alpha} F^k x_k

= e^{-(\alpha(\beta)} F^s E_{\beta} x_s + \sum_{k=0}^{s-1} F^k y_k, \quad \text{where} \quad y_k \in A,
$$
by Lemma 9.2a. Using Corollary 4.1, it follows that $E_0^* x_s = 0$, hence $x_s \in A_{[1]}$. Since by applying a suitable power of $F$ (here we use (a)), we can make any $x_s$ to enter in the last term, this proves (b).

In order to prove (c) note that the subalgebra of $P_{[1]}$ generated by $E, F$ and $K_{[1]}$ is isomorphic to Mat$_4(C)$ (cf. [DC-K]). Hence with respect to this subalgebra, the module $V_{[1]}$ decomposes into a direct sum of $l$-dimensional irreducible submodules. Hence the same is true for $C$ and therefore these exists $x \in C$ such that $E^{l-1} x \neq 0$.

Write $x = \sum_{k=0}^{l-1} F^k z_{x_k}$, where $s \leq l - 1$, $z_s \neq 0$ and $x_k \in A_{[1]}$ (by (b)). Applying $E$, we obtain:

$$E^* z = E^* F^* x_s = \text{const} x_s, \quad \text{where} \quad \text{const} \neq 0.$$  

This proves (c). \qed

\section{Proof of the theorem.} Fix the reduced expression (9.1) of the longest element of $W$, so that $R^+ \setminus R^+ = \{ \gamma_1, \ldots, \gamma_t \}$, where the $\gamma_i$ are defined by (9.2). For $j \in \{ 1, \ldots, l \}$ we let:

$$E_{(j)} = E_{\gamma_j}, \quad F_{(j)} = F_{\gamma_j},$$

$$B_{(j)} = T_{1_{j-1}} \ldots T_{j-1} B_j, \quad P_{(j)} = T_{1_{j-1}} \ldots T_{j-1} P_{(j-1)}$$

$$N_{(j)} = T_{1_{j-1}} \ldots T_{j-1} N_{(j-1)}, \quad \text{etc.}$$

Then Lemma 9.3 holds if the index 1 is replaced by $j$.

Let $V^0$ be an irreducible $U^0_{x}$-module. Note that the ideal of $U^0_{x}$ generated by the $E_0^*$ for $\beta \in R^+ \setminus R'$ acts on $V^0$ nilpotently, hence trivially. Thus $V^0$ is actually a $U^0_{x}$-module.

Let $\tilde{V} = U^0_{x} \otimes_{U^0_{x}} V^0$. We shall show that this is an irreducible $U^0_{x}$-module.

Let $V^i = P_{(i)} \otimes_{B_{(i)}} V^{i-1}$ for $i \geq 1$. Since (by Lemma 9.1b) $B_{(i+1)} \subset P_{(i)}$ we have canonical inclusions:

$$V^0 \subset V^1 \subset V^2 \subset \ldots \subset V^i = \tilde{V}.$$  

Let $A$ be a $U^0_{x}$-submodule of $\tilde{V}$ different from $\tilde{V}$. Then $A \cap V^{i-1} = 0$ since otherwise $A \supset V^0$ and hence $A = \tilde{V}$. Suppose that $A \cap V^{i-1} = 0$. We shall prove that $A \cap V^i = 0$, which proves the irreducibility of $\tilde{V}$. Assuming the contrary, suppose that $C$ is an irreducible $P_{(i)}$-submodule of $A \cap V^i$. Since $N_{(i)}$ acts nilpotently on $V^i$, we conclude (using Lemma 9.2a) that $N_{(i)} C = 0$. Hence it suffices to show that

$$E_{i+1} V^i_{[i+1]} = 0. \quad (10.1)$$

Indeed, by Lemma 9.3c (which can be used due to (9.3)) we deduce from (10.1) that $C \cap V^{i-1} \neq 0$, a contradiction with $A \cap V^{i-1} = 0$.

By Lemma 9.2a, (10.1) is an immediate consequence of

$$V^i_{[i+1]} \subset F^i_{(i)} \ldots F^i_{(1)} V^0, \quad (10.2)$$
which we shall prove by induction. Since \( F(i) \in \mathcal{N}(i+2) \) (by Lemma 9.1b), we have:
\( F(i) V^k_{i+1} = 0 \). Hence \( V^k_{i+1} \subset F^l_{(i)} V^{l-1} \). We now prove by induction on \( k \leq l \) that

\[
V^k_{i+1} \subset F^l_{(i)} \cdots F^l_{(k+1)} V^k.
\]

(10.3)

By the inductive assumption, we may write any \( v \in V^k_{i+1} \) in the form \( v = F^l_{(i)} \cdots F^l_{(k+1)} v_0 \), where \( v_0 \in V^k \). By Lemma 9.1b, \( F(k) v = 0 \), hence

\[
0 = F(k) F^l_{(i)} \cdots F^l_{(k+1)} v_0 = \text{const} \ F^l_{(i)} \cdots F^l_{(k+1)} F(k) v_0,
\]

where \( \text{const} \neq 0 \), by Lemma 3.2 and (9.4). Hence \( F(k) v_0 = 0 \) and therefore \( v_0 \in F^l_{(i)} V^k \) (since we are in an induced module, monomials are linearly independent due to Corollary 4.1). This completes the proof of irreducibility of the \( U_X \)-module \( V \).

Thus (b) is proved since the \( U_X \)-module \( V \) is a non-zero homomorphic image of the irreducible induced module from the \( U^q \)-module \( V' \).

In order to complete the proof of the theorem, we need to show that \( V' \) is a unique irreducible \( U^q_X \)-submodule of \( V \). To show this we introduce a gradation \( V = \bigoplus_{j \in \mathbb{Z}_+} V_j \) by letting \( V_0 = V' \) and \( \deg F(j) = 1, j = 1, \ldots, t \). Due to (9.4), \( F(j) V_j \subset V_{j+1} \). If now \( V'' \) is another irreducible \( U^q_X \)-submodule of \( V \), then obviously, \( V'' \subset \bigoplus_{j \geq 0} V_j \), hence \( V = \bigoplus F^{\ast}_{(i)} \cdots F^{\ast}_{(i)} V'' \subset \bigoplus_{j \geq 0} V_j \), a contradiction. \( \square \)

References


[L] Lustzig, G., Quantum groups at roots of 1, Geom. Ded. 35 (1990), 89–114.


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