

## Vibration: Amplitude and Phase

This manipulative relates to the differential equation controlling a piston driving a spring/mass/dashpot system, as shown on the left window of this manipulative. If you press the “>>>” key you can see the system at work.

(a) Declare in words the significance of the blue curve and the yellow curve; of the blue diamond, the yellow diamond, and of the white line segment joining them; and of the horizontal red line segment (whose length is given by  $t_0$  at the bottom) and the two horizontal white lines on the graphing window.

(b) The motion of the piston is given by  $\cos(\omega t)$  relative to its average position: amplitude 1, circular frequency  $\omega$ . The mass of the spring is taken to be 1. Write  $x$  for the displacement of the mass from equilibrium,  $k$  for the spring constant, and  $b$  for the damping constant. Derive the claimed equation,

$$\ddot{x} + b\dot{x} + kx = k \cos(\omega t).$$

What has to be accounted for is the  $k$  on the right hand side.

(c) This equation usually has a periodic solution, which can be most easily obtained using the complex exponential function. The given equation is the real part of a different ODE, namely

$$\ddot{z} + b\dot{z} + kz = ke^{i\omega t}.$$

As long as  $i\omega$  is not a root of the characteristic polynomial  $p(s) = s^2 + bs + k$ , this equation has as periodic solution the function

$$z_p = \frac{ke^{i\omega t}}{p(i\omega)}.$$

Check this statement. The real part of  $z_p$  is a solution to the original equation, and we will write  $x_p$  for it. It's the only periodic solution, and in fact it's sinusoidal, given by

$$x_p = A \cos(\omega t - \phi).$$

Express  $A$  and  $\phi$  in terms of the polar coordinates of the complex number  $k/p(i\omega)$ .

(d) Now set  $k = 2.00$  and  $b = 0.5$ . Move the  $\omega$  slider at the upper right, and describe what the effects are on the main screen. What is the largest value of the amplitude of the motion of the mass, and at what value of  $\omega$  does this occur? (This is “near resonance.”) When  $\omega$  is small, is the motion of the mass approximately in synch or approximately out of synch with the driving piston? How about when  $\omega$  is large? What happens in between?

(e) Now press the “Bode and Nyquist Plots” key at bottom right. Keep  $k = 2.00$  and  $b = 0.5$ , and move the  $\omega$  slider back and forth again. Notice that if you position the cursor over either of the top right windows you get a readout. These two windows display “Bode plots,” giving the amplitude and phase response to variation of signal frequency. (Actually, in engineering one usually plots  $\log A$  and  $-\phi$  against  $\log \omega$ .) The bottom right window displays a “Nyquist plot.” Watch the effect of moving  $\omega$ . In words, what does it mean for the displayed solution when the value of  $k/p(i\omega)$  is near 1? For what values

of  $\omega$  does this occur (large? small?) What does it mean for the displayed solution when the value of  $k/p(i\omega)$  is over on the other side, pointing west south west so to speak? For what values of  $\omega$  does this occur?

(f) Notice that the near-resonant peak occurs at very nearly the same frequency as when the phase lag  $\phi$  is equal to  $\pi/2$ . Is this an exact relationship? To answer this, first express  $A^{-2}$  as a function of  $\omega$  (with  $k$  and  $b$  fixed), differentiate it and set the derivative equal to zero, to find the value  $\omega_r$  of  $\omega$  at which resonance peaks. What does  $\phi = \pi/2$  mean for the complex number  $k/p(i\omega)$ ? For what value of  $\omega$ , say  $\omega_2$ , does  $\phi = \pi/2$ ? Is  $\omega_2$  equal to, greater than, or less than  $\omega_r$ ? Can you find values of  $k$  and  $b$  for which  $\omega_r$  and  $\omega_2$  are observably different?