

Learning objectives

Phase Line: 1. Interpretation of the phase line: critical points, stable vs unstable, and filling in nonconstant solutions. Horizontal translation carries solutions to other solutions.

2. Implications of the magnitude of $f(y)$ in $y' = f(y)$: rate of transition between critical points.

3. Interpretation of the bifurcation plane: varying a parameter:

- (a) an additive parameter (“harvesting”).

- (b) more complex variation of the system.

Trigonometric Identity: 1. Significance of amplitude A and angular frequency ω of a sinusoidal function. The significance of ϕ in the expression as $A \cos(\omega t - \phi)$.

2. Any linear combination of sinusoidal functions with the same angular frequency is a sinusoidal function with the same angular frequency.

3. More precisely,

$$a \cos(\omega t) + b \sin(\omega t) = A \cos(\omega t - \phi),$$

where the amplitude A is the distance from the origin to (a, b) , and the phase lag ϕ is the angle up from the x -axis.

Complex Roots: The polar representation of complex numbers, and its use in describing roots of complex numbers.

1. Any nonzero complex number has n distinct n th roots, evenly spaced on a circle centered at the origin.

2. As z moves once around the origin, the wheel of roots shifts by one notch.

Complex Exponential: The behavior of the complex exponential e^z .

1. For $z = at$ this is the usual exponential e^{at} . For z purely imaginary, $e^{ibt} = \cos(bt) + i \sin(bt)$ traces a circle.

2. $z \mapsto w = e^z$ carries horizontal straight lines in the z plane to radial rays in the w plane, and vertical straight lines in the z plane to circles in the w plane.

3. $z \mapsto w = e^z$ carries lines through the origin, parametrized by $(a + ib)t$, to spirals in the w plane, parametrized by $e^{(a+ib)t}$.

Tides: 1. Modeling a physical system by a first order ODE.

2. Expression of sinusoidal functions (for example, solutions to an ODE) in terms of amplitude and phase lag; notions of period and time lag.

3. Absent resonance, LTI systems admit unique sinusoidal responses to sinusoidal signals; others differ from it by transients. The sinusoidal system response should be understood in terms of amplitude and phase.

4. Use of complex numbers in deriving a solution to an LTI ODE with sinusoidal signal, and getting from such an expression to amplitude and phase lag information.
5. The periodic system responses to various signal frequencies can be understood in terms of amplitude and phase response curves (expressing these in terms of the circular frequency of the signal).

Questions for use with “Phase Lines”

1. Don't touch anything, for the moment; just describe what you see when the manipulative opens.

Once you have done that, here are some more detailed questions for you.

(a) I claim that the direction field you see before you represents an autonomous ODE. What feature (or features) of the direction field is (or are) relevant to this claim? What features are not relevant?

(b) Can you find the differential equation on the screen? There is a parameter, or constant, called a in the equation. You will notice a slider which is now set so $a = 0$. With $a = 0$, this is called the *logistic* equation, and it represents population growth (or decay) in the presence of a limiting population. Please explain the form of the equation as well as you can, when you think of it as modeling this biological system. A more general form of the logistic equation is

$$y' = ry(b - y).$$

Can you explain the significance of the parameters r and b ?

(c) What does the red line indicate? How about the green line?

(d) When you move the cursor across the graph window, you change the recorded value of y_0 . What do you suppose the significance of y_0 is?

2. Now we'll begin to explore the manipulative.

(a) Click the mouse key while the cursor is positioned somewhere on the graph window. A curve forms. What is it about the curve that supports the belief that it represents a solution to the ODE? Click at several more positions, letting the curve complete after each click. What is the relation between where you have your cursor and where the solution curve begins? Does the horizontal component of the position of the cursor have any significance?

(b) Now click on the “Phase Line” button. A new, very narrow, window opens up. Position the cursor between the lines in the graph window. Describe what appears in the Phase Line window, and explain what its significance is. What happens when you move the cursor around?

(c) Now click somewhere on the graph plane and describe the effect.

(d) Finally, position the cursor to represent the smallest positive value of y_0 that you can and click to generate a solution curve. Estimate the value of y at $t = 8$.

3. Now we'll discuss the parameter a .

(a) Explain, as well as you can, why this a represents the harvest rate, and what it means to say this.

(b) Position the cursor over the slider labeled a , depress the mouse key, and move the mouse. Describe what happens to the graphing plane.

(c) Set $a = 0.20$ and describe what has happened to the critical points. Can you explain this behavior in terms of the population model? What does the region between the red line and the t -axis represent? How about between the red and green lines? Above the green line?

(d) Now position the cursor to represent the smallest value of y_0 that you can which is above the red line, and click to generate a solution curve. Estimate the value of y at $t = 8$. How does this compare with your answer to **2(d)**?

4. To understand what you discovered in **3(d)**, click on the button labeled “DE Graph.”

(a) Describe the significance of the contents of the new window. What is the significance of the red annulus and the green disk? (I am not asking why one is an annulus and the other is a disk—I don’t know the answer to that myself!)

(b) Move the a slider back to $a = 0$. What happens to the curve in the DE Graph window? Compare the maximum value of the graph displayed there now to what it was when $a = 0.20$. Can you relate this to the different answers to **2(d)** and **3(d)**?

(c) Now continue to change the value of a till the red and green spots collide, and then beyond. Describe what happens to the solution curves as the parameter a makes this transition.

5. There is a lot more to explore in this manipulative. Play around with it and describe what you find as you do. All comments are welcome!

6. Do you have any comments about this manipulative and this accompanying guide? Are there some points that are more obscure than others?

Questions for use with “Trigonometric Identity”

1. Open Trigonometric Identity. Don't touch anything, for the moment; just describe what you see when the manipulative opens.

Once you have done that, here are some more detailed questions for you.

(a) Move the top slider, the one controlling A , so that it is set at $A = 2$. (It won't let you go above that, for some reason.) Describe the red curve that appears. Is it correctly represented by the formula you see at upper right?

(b) Now move the next slider down, controlling ϕ . What happens to the red curve? Is this correctly represented by the formula you see at upper right? How is $\cos(\omega t)$ related to $\cos(\omega t - \pi)$?

2. Position ϕ at, say, $\phi = 1$, and leave it. We turn to the bottom window.

(a) Move the a slider. Is the resulting yellow curve correctly represented by its formula? Now move the b slider. Is the resulting blue curve correctly represented by its formula?

(b) Meanwhile, you will have noticed a new curve being formed on the top window, in green. Position the cursor back up in the top window and move it back and forth. Describe what you see and account for it.

3. You are seeing an amazing fact here: any linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ is a sinusoidal function with the same angular frequency, that is, it is of the form $A \cos(\omega t - \phi)$ for some A and ϕ . The goal now is find A and ϕ which match the values of a and b you chose in 2. Play around with A and ϕ till you get close.

It's much harder to go the other way. Pick values of A and ϕ and try to match the red curve with the green one by finding the right values of a and b .

4. Now we'll see a way to make this problem much easier.

(a) Click the lower “parameter plane” button and describe what you see, in connection with your values of a and b . Note the color coding. Then select the upper parameter plane and describe what you see there.

(b) Now pay no attention to the left windows; just move the cursor in the bottom right window till it matches the position of the red marker in the top right window. Do these values of a and b do the trick?

Express as clearly as you can the principle being illustrated here. You may want to make reference to the cosine difference formula

$$A \cos(\omega t - \phi) = A \cos(\phi) \cos(\omega t) + A \sin(\phi) \sin(\omega t).$$

5. Play around with the tool some more.

Do you have any comments about this manipulative and this accompanying guide? Are there some points that are more obscure than others?

Questions for use with “Complex Roots”

1. Open Complex Roots. Don't touch anything, for the moment; just describe what you see when the manipulative opens.

Once you have done that, here are some more detailed questions for you.

(a) The cyan marker can be grasped and moved around. Do this and describe the effect. Greater precision can be achieved by using the “argument” and “modulus” sliders. Move them around and describe what happens to the cyan marker. Describe the significance of the “modulus” and the “argument.”

(b) It's claimed that the cube roots of the complex number marked in cyan are given by the three yellow markers, and their (approximate) values are recorded. To check this, set the cyan marker at $z = 8 + 0i$. What are the resulting values of the three yellow-marked points? The three cube roots appear to cut the circle in equal thirds. Explain this using the principles of multiplication of complex numbers:

moduli multiply, arguments add

What are the arguments of the three cube roots, exactly? Use some geometry of triangles to figure out what the exact values of the two nonreal cube roots.

(c) Now activate the “argument” slider, and move it all the way from $-\pi$ to $+\pi$, so that the cyan marker moves through a full 2π radians. What happens to the cube roots? What fraction of a complete revolution do they make? At π , the configuration has returned to where it was at $-\pi$; but have the roots returned to their original position? How many radians has each one moved?

(d) Select the “angle” key and move the cyan marker around. Describe what the display at right is telling you. Now position the cyan marker at $z = 0 + i$. The picture may be clearer if you hit “Zoom.” Describe the three cube roots of i , and why they are cube roots of i .

2. With this experience, what is your guess about the positions of the seventh roots of 1? Check out your intuition, using this tool. How far will the circle of seventh roots rotate as the cyan marker rotates all the way around the circle?

3. Play around with the tool some more.

Do you have any comments about this manipulative and this accompanying guide? Are there some points that are more obscure than others?

Questions for use with “Complex Exponentials”

1. Open Complex Exponentials. Don't touch anything, for the moment; just describe what you see when the manipulative opens.

Once you have done that, here are some more detailed questions for you. This tool is about the complex exponential function, which sends the complex number z to the complex number e^z . This is a little hard to grasp because both z and e^z are complex numbers, so each is represented by a point on the plane. The left plane houses z , the right e^z .

(a) When the tool opens, the cyan point in the left window is at $a + bi = .5 + 5i$. Move the cyan marker around. Pay no attention to the right window for the moment! Please provide a parametrization of the straight line which passes through the origin at $t = 0$ and through a point $a + bi$ on the complex plane at $t = 1$. (This expression will of course involve $a + bi$.)

(b) When you click and release on the left screen, this line is drawn out slowly. Can you explain why it is drawn more slowly when $a + bi$ is near to the origin than when it is far away from the origin?

2. (a) Use the sliders to position $a + bi$ at $1.00 + 0.00i$. The yellow straight line on the left screen is thus horizontal, and takes on only real values. Peek at the right screen: what do you see? What SHOULD you see? Notice that the red point is always at 0 in the left screen and 1 in the right screen. Is this sensible? What is the exact position of the cyan point in the right screen now?

(b) For purely imaginary values of z , say $z = ibt$, e^z has the form

$$e^{ibt} = \cos(bt) + i \sin(bt).$$

What curve will this traverse as t changes? Check this out, by using the sliders to set $a = 0$. (You may have discovered by now that in many of these manipulatives clicking on the hashmarks will set the slider value exactly.) Then push the b slider from 0 up to 8. Describe what you see on the right screen, and try to account for it.

3. We will now experiment with values of $a + bi$ which are neither real nor complex. This is controlled by

$$e^{(a+ib)t} = e^{at}(\cos(bt) + i \sin(bt)).$$

(a) When $a + bi$ is in the northeast quadrant, what happens to the curve $e^{(a+ib)t}$? Explain this as clearly as you can.

(b) Predict what the curve on the right screen will look like if we pick $a + bi = -.5 + 10i$ on the left hand screen. Will it move in or out? Clockwise or counterclockwise? Check this out.

(c) Where should you position the cursor to produce a spiral that goes out counterclockwise?

4. (a) Use the slider to set $a = .5$, and vary the b slider. A deep blue circle appears on the right window, and the cyan point moves along it. Can you explain why this is so?

(b) Use the slider to set $b = 4$, and vary the a slider. What appears on the right hand window? Can you explain what is happening here?

5. Play around with the tool some more.

Do you have any comments about this manipulative and this accompanying guide? Are there some points that are more obscure than others?