Learning objectives

Phase Line: 1. Interpretation of the phase line: critical points, stable vs unstable, and filling in nonconstant solutions. Horizontal translation carries solutions to other solutions.

2. Implications of the magnitude of $f(y)$ in $y' = f(y)$: rate of transition between critical points.

3. Interpretation of the bifurcation plane: varying a parameter:

- (a) an additive parameter ("harvesting").

- (b) more complex variation of the system.

Trigonometric Identity: 1. Significance of amplitude A and angular frequency ω of a sinusoidal function. The significance of ϕ in the expression as $A\cos(\omega t - \phi)$.

2. Any linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ is a sinusoidal function with angular frequency ω.

3. More precisely,

$$a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi),$$

where the amplitude A is the distance from the origin to (a, b), and the phase lag ϕ is the angle up from the x-axis.

Complex Roots: The polar representation of complex numbers, and its use in describing roots of complex numbers.

Complex Exponential: The behavior of the complex exponential; illustrated by the curve $e^{(a+bi)t} = e^{at}(\cos(bt) + \sin(bt))$.

Tides and Amplitude and Phase Lag: 1. Modeling a physical system by a first order ODE.

2. Expression of sinusoidal functions (for example, solutions to an ODE) in terms of amplitude and phase lag; notions of period and time lag.

3. Absent resonance, LTI systems admit unique sinusoidal responses to sinusoidal signals; others differ from it by transients. The sinusoidal system response should be understood in terms of amplitude and phase.

4. Use of complex numbers in deriving a solution to an LTI ODE with sinusoidal signal, and getting from such an expression to amplitude and phase lag information.

5. The periodic system responses to various signal frequencies can be understood in terms of amplitude and phase response curves (expressing these in terms of the circular frequency of the signal).

6. In the second order case, the system reponse is in synch with the signal when the signal frequency is much less than the natural frequency of the system, and out of synch when it is much bigger.

Fourier Coefficents: 1. Visualizing scaling and adding functions.

2. Symmetry properties of sine and cosine: sine is odd, cosine is even; $\sin(kt)$ is odd about $\pm\pi/2$ if k is odd, even if k is even; $\cos(kt)$ is odd about $\pm\pi/2$ if k is even, even if k is odd.

3. Fourier coefficients are determined by optimally approximating the desired function (in a least squares sense, but understanding this in detail is not a goal).

4. The approximation is faster for continuous functions than for discontinuous ones.

5. Orthogonality: there is a single number, depending upon the Fourier series and the function we are trying to approximate, which achieves a minimum exactly when the Fourier coefficients are correct, independent of the order in which they are selected.

Convolution: Looking Forward: 1. A system responses can be thought of as a sum of responses to earlier short segments of the signal. The response to an isolated event is captured by the "weight function."

2. The expression of this idea as an integral, the convolution integral.

3. If the weight function is of an appropriate type, the system can be modeled by an ODE, having the weight function as a solution with specific simple initial values. [One could mention the notion of impulse response here, but this is not explictly part of the manipulative.] The convolution expresses the solution of an inhomogeneous LTI equation, with rest initial conditions, directly in terms of the signal and the weight function.

4. The convolution product is commutative and associative. [This can be seen experimentally in this tool, but it seems accidental.]