

Questions for use with “Convolution: Looking Forward”

Preliminaries: This manipulative illustrates one perspective on the “convolution integral.” It’s useful to keep the following image in mind. We are interested in the accumulation of a certain chemical, X, in a lake. X is known to decay, or evaporate out of the lake, exponentially over time, so that of A units of X at time u there are $Ae^{-a(t-u)}$ units remaining at time $t > u$. More generally, let’s write $w(t)$ for the decay rate; it might be some function other than e^{-at} . X is being dumped into the lake at some variable rate, say $f(t)$. The final assumption is that the amount is zero at $t = 0$. Write $x(t)$ for the number of units of X in the lake at time t .

The “decay rate” $w(t)$ is called the *weight function*, and the rate of input $f(t)$ is called the *signal*.

Do you agree that this information should determine the function $x(t)$?

1. Don’t touch anything, for the moment; just describe what you see when the manipulative opens.

Once you have done that, here are some more detailed questions for you.

(a) You’ll be able to select from several different choices of weight function and signal. When the tool opens, $f(t) = w(t) = 1$. Explain what this signifies in terms of the model offered above. This weight function actually *is* an exponential; what’s a ? With these choices, what’s $x(t)$? It’s displayed in the upper window. Do you agree with what you see?

To watch the level of X build up, click on some time t .

(b) The default setting has “step size” $1/2$, meaning that the additions to the lake are represented as occurring discretely, every $1/2$ time unit. The weight is a step function with height 1; what is the height of the layer it contributes? You can highlight this layer by clicking on either of the graphing windows.

(c) Now select step size $1/4$ and click on a time value to watch X accumulate. Comment.

2. Now select the weight function e^{-at} .

(a) To see the shape of the weight function you can click on the bottom graphing window. Click successively at $t = 1/4, 1/2, 3/4, 1$. Explain the significance of the regions which appear, outlined in cyan, in the upper window. Can you relate the vertical width of those regions to the step size? You may want to change the step size to check your idea. Click at $t = 12$ to see the whole picture.

(b) Focus on the value of x at $t = 12$. It is a sum of quantities of X which dumped into the lake at various times $u < t$; but these quantities have decayed since they were put

into the lake. How much was put into the lake between time u and time $u + du$ (where du is some small amount of time)? By what factor has that amount decayed by time t ? Express the sum of these decayed contributions as an integral, and evaluate the integral. Does it correspond to the graph in the upper window?

(c) How will this expression change if we replace the weight function $w(t) = e^{-at}$ with a general weight function $w(t)$?

(d) How will *this* expression change if we replace the signal $f(t) = 1$ by a general signal $f(t)$?

You have just written down the *convolution integral*, or the convolution $x = f * w$.

3. Explore some of the other weights and signals, and see that they make sense to you. Two of the weights grow rather than shrink. Maybe instead of a chemical decaying we should think of a species of fish, or bacteria, propagating. Talk about what you see.

4. In the case $w(t) = e^{-at}$, but with arbitrary signal $f(t)$, we can think of the amount of X in the lake as being controlled by a first order ODE: at time $t + \Delta t$ the amount of X is

$$x(t + \Delta t) = x(t) - ax(t)\Delta t + f(t)\Delta t$$

so putting the $x(t)$ on the left, dividing through by Δt , and taking the limit $\Delta t \rightarrow 0$, we get the ODE

$$\dot{x} + ax = f(t)$$

The weight function satisfies this equation with $f(t) = 0$ with initial condition $w(0) = 1$. Explain why we have now shown that the convolution product $f * w$ is the solution to $\dot{x} + ax = f(t)$ with $x(0) = 0$.

5. The weights $w(t) = 1$ and $w(t) = e^{-at}$ both have value $w(0) = 1$, while the others have $w(0) = 0$ and $w'(0) = 1$. They differ in another way too: the first two satisfy a first order ODE ($\dot{w} = 0$ for $w(t) = 1$, and the natural decay equation $\dot{w} + aw = 0$ for $w = e^{-at}$), while the others satisfy second order equations. For example, check that $w(t) = te^{-at}$ satisfies the initial value problem

$$\ddot{w} + 2a\dot{w} + a^2w = 0, \quad w(0) = 0, \quad \dot{w}(0) = 1.$$

This is a typical unforced damped oscillator. The convolution $f * w$ gives the solution to the initial value problem

$$\ddot{x} + 2a\dot{x} + a^2x = f(t), \quad x(0) = 0, \quad \dot{x}(0) = 0.$$

Can you explain why?

6. Do you have any comments about this manipulative and this accompanying guide? Are there some point that are more obscure than others?