Smith's talk
(aug 12)

Part I (of notes - "intuition")

Wed. Aug 12 2009

Jeff Smith

"Moduli spaces of homotopy G spheres"

a htpy G-sphere -
G-space X s.t. X is w. equiv to some sphere.

Joint work w/ Jesper Grodal -
tried to classify all homotopy G-spheres for finite groups.

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Question: what to take as equivalences?

X, Y ∈ Htpy G Sph. If their Borel constructions are equiv. as spaces over BG, (weakly)

$X \times EG$ is spherical fibration
$\downarrow$
BG ⇒ actually

Want to compute

$[BG, BAut\, S^n]$ //

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Moduli space of htpy G-spheres

$\mathcal{M}_G = \coprod_{n \geq 0} Map[BG, \cancel{BAut}\, S^{n-1}]$ $Bhaut$

space of all ~~spatic~~ spherical fibrations w/ fiber S^{n-1} (?)

($S^{-1} = \emptyset$ (new name for empty set))

$$\begin{array}{ccc} haut(S^n) & \longrightarrow & Map(S^n, S^n) \\ \downarrow & & \downarrow \\ (\pm 1) & \longrightarrow & \pi_0 Map(-) = \mathbb{Z} \end{array}$$

grouplike topological monoid of homotopy self equiv. of S^n.

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"n-1" for convenience: b/c of first observation: (more like, as monoid)

M_G is actually a topological monoid under join

join: $S^{n-1} * S^{m-1} = S^{n+m-1}$

$\Rightarrow$ $\mathrm{Bhaut}\, S^{n-1} \times \mathrm{Bhaut}\, S^{m-1} \to \mathrm{Bhaut}\, S^{n+m-1}$

$(\text{unit} = \emptyset = S^{-1})$

Exercise: every spherical fibration is ~~w.e.~~ equi. to a sphere

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G - finite

S^G := Obj: htpy G-spheres
Mor: $X \xrightarrow[G]{f} Y$, f is equiv. and $\to$ an equiv?

$M_G = N S^G$

$\Rightarrow M_G = \coprod_{[X]} \mathrm{Bhaut}_G(X)$

Theorem

(Kan, Dwyer, etc)

M_G $\quad \pi_0 M_G$ — abelian monoid

$\pi_0 M_G$ has been computed.

$\pi_0 \mathrm{Bhaut}\, X \cong [X, X]_G$
and can compute

π_i Bhaut $X \cong$ torsion $\oplus$ torsion free
finite gp, else a mystery; at most 1 a copy of $\mathbb{Z}$ / computable; some copies of $\mathbb{Z}_p$, $p \mid |G|$

depends on where there is a copy of $\mathbb{Z}$:
Rational homotopy type of Bhaut S^n
- can look inside Sullivan for this

Q: What is spectrum for $S^{-1} = \emptyset$
don't know, but should have something to do w/ G.

Corollary: for G - a p-gp, every homotopy group is linear.

How to prove such things?
Framework:
$\boxed{\text{Borel equivariant homotopy theory}}$
(Look at homotopy fixed points instead of actual fixed points)
This becomes a practical subject after the work of Lannes.

How to study? One prime at a time - must fracture it. - do padic Borel ~~homotopy~~ equiv htpy theory

How to prove these results?

Need: (p-adic) (equivariant

Fracture square

$\Rightarrow$ p-adic Borel equiv. htpy theory.

The right way to think abt.

Borel G-spaces as local objects

in orbit category

$f(\mathcal{O}_G)^{op}$

Main technical result: [hyper Sullivan conjecture]

Recognition theorem for

local objects

(Sullivan conjecture - is a special

diagram local $X^{\mathcal{O}_G^{op}} \longrightarrow X^{\mathcal{O}_G^{op}}$

ANTONIO LURO

Jeff Smith's talk on ~~class~~ moduli spaces of homotopy G spheres

August 12, 2009, Wednesday

Borel equiv htpy theory vs classical equiv htpy th.

Classical equiv

a map of G spaces

$X \xrightarrow[G]{} Y$ is a classical weak equiv. if map on fixed points is a weak equiv. of spaces for all $H < G$.

Equiv. map is a Borel equivalence if map on Borel constructions is a weak equiv.

(this will imply $f^{hH} : X^{hH} \longrightarrow Y^{hH}$ is a weak equiv.)

(so care about homotopy fixed points)

(defn of hfp : X^{hG} = space of sections $\begin{array}{c} X \times_G EG \\ \downarrow \\ BG \end{array}$)

<u>Fracture square</u> (for X - nilpotent)

~ a homotopy pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\nwarrow h"] & \prod X_{\hat{p}} \arrow[d] \\ (X)_{\mathbb{Q}} \arrow[r] & \left(\prod X_{\hat{p}}\right)_{\mathbb{Q}} \end{tikzcd}$$

Must somehow make an <u>equivariant</u> htpy pullback square like this for X - a G space

Rest of time: p-adic Borel equiv. htpy theory.

Homotopy of p-adic spaces w/ a G-action.

$\equiv$ htpy theory of spaces over BG w/ a p-adic htpy fiber

$\boxed{\text{Notation}}$

$\mathcal{O}_p G$ - p-orbit category Obj: G/D, D - p subgp of G (D for defect)

$\mathcal{S}_{\hat{p}}$ - p-adic spaces

Define a Functor

$$\mathrm{Fix}: \; G\text{-spaces} \longmapsto \mathcal{S}_{\hat{p}}^{(\mathcal{O}_p G)^{op}}$$

(do this instead of $\mathrm{Map}(G, -)$ to get <u>homotopy</u> fixed points)

$$X \longmapsto \left(G/D \longrightarrow \mathrm{Map}(EG \times_G - , X)_{\hat{p}} \right.$$

Have two functors

$$p\text{-adic } G\text{-spaces} \; \underset{\mathrm{Eval}(G/e)}{\overset{\mathrm{Fix}}{\rightleftarrows}} \; \mathcal{S}^{(\mathcal{O}_p G)^{op}}$$

(top is right adjoint)

these are htpy adjoint

$\mathrm{Ev} \circ \mathrm{Fix} = id$

There is a localization functor $L: \mathcal{S}^{-} \longrightarrow \mathcal{S}^{-}$ $\mathrm{Fix} \circ \mathrm{Ev}$ is a <u>localization</u>

htpy theory of local objects $\equiv$ htpy theory of Borel G-spaces.

Recognition theorem

$$\mathcal{F}(\mathcal{O}_p G) \longrightarrow \mathcal{C}_V^{(\mathcal{O}_p G)^{op}} = \mathcal{C}(V^{\mathcal{O}_p G^{op}})$$

V = vector sp / $\mathbb{F}_p$

$\mathcal{C}_V$ = chain complexes of vector sp

$\mathcal{A}(p)$ - abelian cat $(\mathcal{O}_p G) \to V$

$\mathcal{A}(p)$ - ab cat w/ enough of everything

D - a diagram

If $C_* D$ is quasiisomorphic to a bounded complex of projectives,

then D is local

(in particular, $D(G/e)^{hi} = D(G^1(D)$)

this is exactly the Sullivan conjecture.

Remark

Fusion condition comes up when discussing realizable dimension functions

$$d : (\mathcal{O}_p G) \to \mathbb{N}$$

if $d(G/D_2) < d(G/D_1)$, $d \geq 1$, then

$$[X, X]_G \cong \underset{\substack{\uparrow \\ \text{Burnside ring}}}{\left(A(G)\right)}_{\hat{I_p}}, n$$