

Wed. Aug 12 2009

Jeff Smith

"Moduli spaces of homotopy G-spheres"

A htpy G-sphere -
G-space X st. X is w. equiv
to some sphere.

Joint work w/ Jesper Knudsen
tried to classify all
homotopy G-spheres
for finite groups.

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Smith's talk
(Aug 12)

Part I (of notes -
"intuition")

Question: what to take
as equivalences?

$X, Y \in \text{Htpy } G \text{ sph.}$ If
their Borel constructions
are equiv. as spaces over BG
(weakly)

$X \times EG$ is spherical fibration
 \downarrow
 $BG \Rightarrow$ actually

Want to compute

$[BG, BAut S^n]$

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Moduli space of htpy G-spheres

$M_G = \coprod_{n \geq 0} \text{Map}(BG, BAut S^n)$

space of all
spherical fibrations w/ fiber S^{n-1}

($S^{-1} = \emptyset$ (new name for empty set))

$\text{haut}(S^n) \rightarrow \text{Map}(S^n, S^n)$

\downarrow
 $(\pm 1) \rightarrow \text{Map}(-) = \mathbb{Z}$

group-like topological monoid
of homotopy self-equiv.
of S^n

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"n-1" for convenience: b/c of first observation: *(more like the monoid)*

M_G is actually a topological monoid under join

$$S^{n-1} * S^{m-1} = S^{n+m-1}$$

$$\text{Bhaut } S^{n-1} \times \text{Bhaut } S^{m-1} \rightarrow \text{Bhaut } S^{n+m-1}$$

(unit = $\emptyset = S^{-1}$)

Exercise: every spherical fibration is equiv. to a sphere

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G - finite

$S^G := \text{Obj: htpy } G\text{-spheres}$

Mor: $X \xrightarrow{f} Y$

f is equiv.

and \rightarrow is equiv?

$$M_G = WS^G$$

$$\Rightarrow M_G = \coprod [X] \text{ Bhaut}_G(X)$$

Theorem $\nearrow [X]$

(Kan, Dwyer, etc)

M_G $\text{Th} M_G$ - abelian monoid

$\text{Th} M_G$ has been computed.

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$\text{Th} \text{Bhaut } X \subseteq [X, X]_G$
and can compute

II: $\text{Bhaut } X \subseteq \text{torsion} \oplus \text{torsion free}$
finite gp, etc a mystery computable at most 1 a copy of \mathbb{Z} some copies of \mathbb{Z}/p

where there is a copy of \mathbb{Z} :

Rational homotopy type of $\text{Bhaut } S^q$

- can look inside Sullivan for this

Q: What is spectrum for $S = \emptyset$
don't know, but should have something to do w/ G .

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Corollary: for G -a-p-gp, every homotopy group is linear.

How to prove such things?

Framework:

Borel equivariant homotopy theory (look at homotopy fixed points instead of actual fixed points)

This becomes a practical subject after the work of Lannes.

How to study? One prime at a time - must fracture it.

do p-adic Borel equiv htpy theory

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How to prove these results?

Need p -adic (equivariant)

Fracture square

\Rightarrow p -adic Borel equiv. htpy theory

The right way to think abt

Borel G -spaces as local objects

in orbit category

$\mathcal{O}_{G/H}$

Main technical result:

Recognition theorem for

local objects

[hyper
Sullivan
conjecture]

(Sullivan conjecture -- is a special

diagram local $X^{hG} \rightarrow X^{oG}$

Jeff Smith's talk on ~~class~~
moduli spaces of homotopy G spheres

August 12, 2009,
Wednesday

Borel equiv. h. theory
vs
classical equiv. h. theory

Classical equiv

a map of G spaces

G.

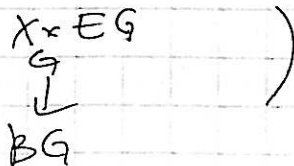
$X \xrightarrow{G} Y$ is a classical weak equiv. if map on fixed points is
a weak equiv. of spaces for all $H < G$.

Equiv. map is a Borel equivalence if map on Borel constructions
is a weak equiv.

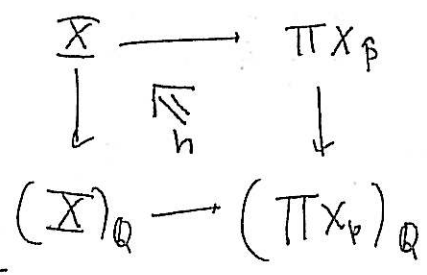
(this will imply $f^{hH} : X^{hH} \rightarrow Y^{hH}$ is a weak equiv.)

(so care about homotopy fixed points)

(defn of hG : $X^{hG} =$ space of sections)



Fracture square (for X -nilpotent)
 - a homotopy pullback



Must somehow make an equivariant htpy pull back square like this for X a G -space

Rest of time: p-adic Borel equiv. htpy theory.

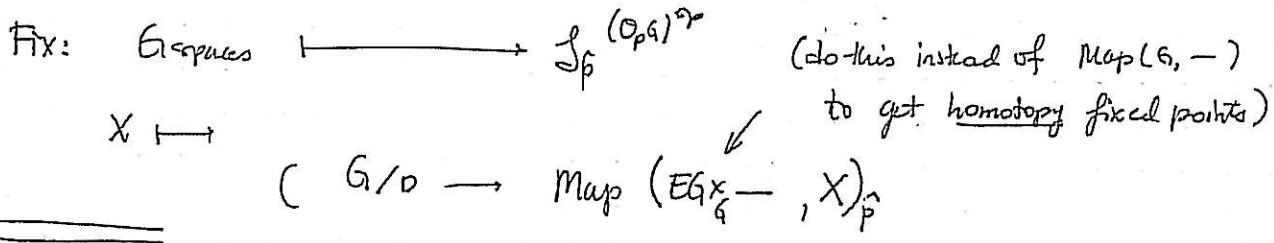
Homotopy of p-adic spaces w/ a G -action.

≡ htpy theory of spaces over BG w/ a p-adic htpy fiber

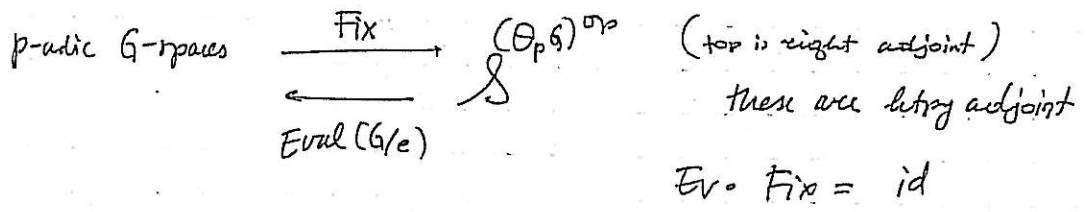
Notation

$\mathcal{O}_p G$ - p-orbit category Obj: G/D , D - p-subgrp of G
 (D for defect)

\mathcal{S}_p - p-adic spaces
 Define a Functor



Have two functors



There is a localization functor $L: \mathcal{S} \rightarrow \mathcal{S}^-$ $\text{Fix} \circ Ev$ is a localization

htpy theory of local objects \equiv htpy theory of Borel G -spaces.

Recognition theorem

$$\mathcal{S}(\mathcal{O}_p G) \longrightarrow \mathcal{C}_V^{(\mathcal{O}_p G)^{op}} = \mathcal{C}(V^{\mathcal{O}_p G^m})$$

$V = \text{vector } \mathfrak{g}_0 / \mathbb{F}_p$

$\mathcal{C}_V = \text{chain complexes of vectors, } \mathfrak{g}_p$

$\mathcal{A}(p) - \text{abelian cat } (\mathcal{O}_p G) \rightarrow V$

$\mathcal{A}(p) - \text{ab cat w/ enough of everything}$

$D \Leftarrow \text{a diagram}$

If $C \in D$ is quasiisomorphic to a bounded complex of projectives,
then D is local

$$(\text{in particular, } D(G/e)^{nil} = D(G^1/D))$$

this is exactly the Sullivan conjecture.

Remark

Fusion condition comes up when discussing
realizable dimension functions

$$d: (\mathcal{O}_p G) \rightarrow \mathbb{N}$$

if $d(G/D_2) < d(G/D_1)$, $d \geq 1$, then

$$[X, X]_G \cong \left(A(G) \right)_{\mathbb{I}_p}^{\wedge, n}$$

↑
Burnside
ring