SOME REMARKS ON THE KERVAIRE INVARIANT PROBLEM FROM THE HOMOTOPY POINT OF VIEW

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The object of this note is to discuss some results which were obtained in an effort to settle the Kervaire invariant conjecture.

There is a secondary cohomology operation based on the Adem relation which expands $Sq^{2^j}Sq^{2^j}$. Call this $\varphi_{j,j}$, after Adams.

CONJECTURE A. There exists a two cell complex $S^n \cup e^{n+N+1}$ $(N = 2^{j+1} - 2)$ with $\varphi_{j,j}$ nonzero. We call any such element θ_j. (Note that this defines only a coset.)

Browder has shown that this conjecture is equivalent to the existence of a framed manifold with nonzero Kervaire invariant.

There are many statements which imply the conjecture. Most are conjectured to be equivalent. Let us begin with the simplest.

Suppose $X = S^0 \cup_{2^l} e^1$, that is, the space in the stable category which represents $\Sigma^{-1}RP^2$.

THEOREM 1. *An element of Hopf invariant* 1 *in* $\Pi_{2^{j+1}-1}(X)$ *implies* A *in* dim $2^{j+1} - 2$.

PROOF. An element of Hopf invariant one implies a three cell complex so that $Sq^{2^{j+1}} \neq 0$. Adams has shown that $Sq^{2^{j+1}} = \sum a_{i,k,j}\varphi_{i,k}$. We apply this to the Spanier-Whitehead dual of the complex and conclude $\varphi_{j,j} \neq 0$ and $a_{j,j,j} = Sq^1$.

COROLLARY 2. *If* $[\iota_N, \iota_N] = 2\alpha$, $N = 2^{j+1} - 1$, *then* $\alpha = \theta_j$.

We can use Theorem 1 to try and construct the θ_j's inductively. Indeed,

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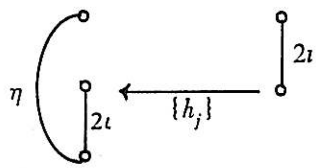
[1] These notes are based on the joint work of M. G. Barratt and M. E. Mahowald.

suppose there is an element in dimension $2^j - 1$ of Hopf invariant 1. Let us call it $\{h_j\}$.

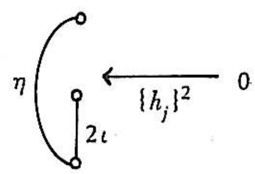
PROPOSITION 3. $2\{h_j\} = \eta\theta_{j-1}$.

This follows immediately from Theorem 1 and a standard relationship in the homotopy of X.

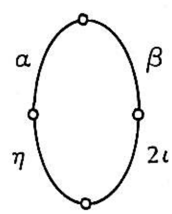
Thus $\{h_j\}$ admits an extension by 2 modulo η. This gives a map



This clearly defines a map



and a four cell space



with attaching maps $\alpha = \theta_{j-1}^2$ and $\beta = \langle\theta_{j-1}, 2\iota, \theta_{j-1}\rangle$ such that $\varphi_{j,j}$ is nonzero from the bottom to the top cell. Summarizing, we have shown:

THEOREM 4. *There is a null-homotopy defined on S^{2^j} of*

$$\eta\theta_{j-1}^2 + 2\iota\langle\theta_{j-1}, 2\iota, \theta_{j-1}\rangle$$

which carries a $\varphi_{j,j}$.

COROLLARY 5. *If there is a non-$\varphi_{j,j}$ carrying null-homotopy of*

$$\eta\theta_{j-1}^2 + 2\iota\langle\theta_{j-1}, 2\iota, \theta_{j-1}\rangle$$

then θ_j exists.

Relevant to this is the following canonical relation.

PROPOSITION 6. *If* $\alpha \in \Pi_k^S$, $2\alpha = 0$, $k \equiv 2 \pmod 4$, *then* $\langle\alpha, 2\iota, \alpha\rangle = 0$.

Thus Corollary 5 has a weaker version.

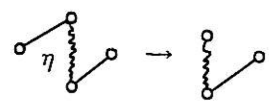
COROLLARY 5′. *If there is a non-*$\varphi_{j,j}$ *carrying null-homotopy of* $\eta\theta_{j-1}^2$, *then* θ_j *exists.*

Thus we have the first part of the following theorem.

THEOREM 7. *If* θ_{j-1} *exists,* $2\theta_{j-1} = 0 = \theta_{j-1}^2$ *then* θ_j *exists and* $2\theta_j = 0$.

The second part follows by a similar analysis.

Another proof of Theorem 4 is based on the smash product. Let $S^{2^j-1} \to X$ represent $\{h_j\}$. Then $S^{2^j-1} \wedge S^{2^j-1} \to X \wedge X$ represents $\{h_j^2\}$ and this gives a map

$S^{2^{j+1}-2} \longrightarrow$ 

as before. This proof is a version of the central idea of our approach. The philosophy is to use the existence of $\theta_{j'}^{j'+j}$ and apply a functorial construction which hopefully gives θ_j. The Γ construction discussed earlier by Barratt is an example. It contains the "quadratic" construction and higher symmetries. In particular, explicit construction of the 30-manifold using $\mathscr{S}_4$, the symmetric group on four letters, has been given.

Milgram, using $\mathscr{S}_4$ symmetries, has proved the following:

THEOREM 8 [MILGRAM]. *With the hypothesis of Theorem* 7, θ_{j+1} *exists.*

REMARK. It can be shown that $\theta_4^2 = 0$ and thus Milgram's theorem implies θ_6 exists.

A more delicate argument about θ_{j-1} shows

PROPOSITION 9. $\langle\theta_{j-1}, 2\iota, \theta_{j-1}\rangle_n = [\iota_n, \beta_{j-1}]$, *where* $n = 2^{j+1} - \varphi(j-1) - 1$, β_{j-1} *is the generator of the* im J *in* stem $\varphi(j-1) - 1$ *and* $\varphi(j-1)$ *is the Adams function.*

COROLLARY 10. $\Sigma^{-\varepsilon}2\langle\theta_{j-1}, 2\iota, \theta_{j-1}\rangle_n = [\iota_{2^{j+1}-\varphi(j)-1}, \beta_j]$ *where* $\varepsilon = \varphi(j) - \varphi(j-1)$.

Thus

THEOREM 11. *If* θ_{j-1} *exists and has order* 2 *and* θ_j *exists,* θ_j *appears on the* $2^{j+1} - \varphi(j)$ *sphere with Hopf invariant* β_j.

There is a slightly less direct approach. First observe that there is a map $\lambda: RP \to S^0$ in the stable category. λ on each cell is the Whitehead product. To be precise, there is a map $\Sigma^n P^{n-1} \xrightarrow{\lambda_n} S^n$ and $S^{2n-1} \xrightarrow{\Sigma^n a_n} \Sigma^n P^{n-1}$ where a_n is the natural map $S^{n-1} \to P^{n-1}$. The composite $\lambda_n \Sigma^n a_n = [\iota_n, \iota_n]$. Thus

PROPOSITION 12. *If* $\Sigma^n a_n$ *can be halved for* $n = 2^{j+1} - 1$ *then* θ_j *exists.*

There is a similar statement for CP and QP.

Consider the situation with just a single suspension. We have the following fibration:

$$P * P \to \Sigma RP \to K(Z_2, 2).$$

PROPOSITION 13. *If there is a map $f: S^{2^{j+1}-1} \to P * P$ so that*

$$f^*(\alpha^{2^j - 1} * \alpha^{2^j - 1}) \neq 0$$

then θ_j exists.

A weaker version is also true.

PROPOSITION 13′. *If there is a stable map $f: S^{2^{j+1}-2} \to P \wedge P$ so that*

$$f^*(\alpha^{2^j - 1} \wedge \alpha^{2^j - 1}) \neq 0$$

then θ_j exists.

Even weaker versions than this are possible.

PROPOSITION 13″. *Let v_j be a cohomology class in SO which transgresses to w_{2^j}. Then $v_j \otimes v_j$ being spherical in $SO * SO$ or in $S(SO \wedge SO)^*$ implies Conjecture* A.

Another approach stems from the effort to construct large brackets.

THEOREM 14 (HOFFMAN). *If $\langle \sigma, 2\sigma, 2\sigma, \ldots, 2\sigma, \sigma \rangle$ can be defined then θ_j is in it.*

This is verified for $j = 4$. There is a family of spaces X_k which are defined by identifying particular subspaces of $\Lambda^k(S^8 \cup_\sigma e^{16})$. The cell structure looks like

This shows $0 \in \langle \sigma, 2\sigma, \ldots, (k-1)\sigma, k\sigma \rangle$.

THEOREM 15. *If the bracket $\langle \sigma, 2\sigma, \ldots, (2^{j-2} - 1)\sigma, 2^{j-3}\sigma \rangle$ can be formed then θ_j is in it.*

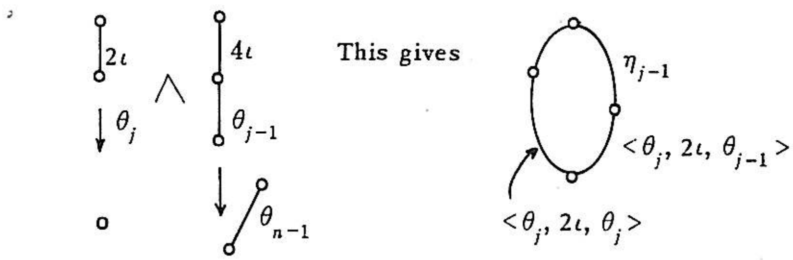
This requires a mild interpretation because $16\sigma = 0$, but it is not hard to see what should be done.

A paraphrase of Theorem 15 is the question of whether the attaching map in the construction X_k can be halved.

Another amusing approach:

THEOREM 16. *If $2\theta_{j-1} = 0 = 2\theta_j$, and $\langle \theta_{j-1}, 2\iota, \theta_j \rangle = 0$ then θ_{j+1} exists.*

PROOF. Take



This construction is $\varphi_{j+1,j+1}$ carrying.

CONJECTURE. $\langle \theta_j, 2\iota, \theta_{j-1} \rangle = [\iota_n, \beta_{j-1}]$ where $n = 2^j + 2^{j-1} - \varphi(j-1)$. (Although $\langle \theta_4, 2\sigma, \sigma \rangle = 0$, it is not known whether $\langle \theta_4, 2\iota, \sigma\sigma \rangle = 0$.)

The strongest evidence for the existence of the θ_j's is

THEOREM 17. *There exists a sequence of integers n_i such that $2^i - 2 < n_i \leqq 2^{i+1} - 2$ and elements $\alpha_i \in \Pi^S_{n_i}$. If $n_i = 2^{i+1} - 2$ then the constructed elements are θ_i.*

PROOF. Simply stated, the α_i are the stable Hopf invariants of the β_i in stem $2^{i+1} - \varphi(i) - 1$. To be more precise consider the $X_{2^j - r}$ constructing. The attaching map of the cell in dimension 2^{j+2} can be halved at least through the $2^{j+2} - 2^j$-skeleton. If it can be halved through the $8 + 2^{j+1}$-skeleton then we have θ_j, otherwise there is an obstruction. We call this obstruction α_j.

Another amusing result is the following.

Let $Y \to S^0 \to K(Z, 0)$ define Y. The map $\lambda: P \to S^0$ lifts to Y.

REMARK. Y/P has cohomology which is free over Sq^1 and Sq^2. Thus as far as *bo* homology is concerned, P and Y are equally interesting.

Consider the spectrum Im J defined by the fibration

$$\text{Im } J \to BO[8k, \ldots] \to BO[8k+4, \ldots].$$

THEOREM 18. $\Pi_i(P \wedge \text{Im } J) =$

$i =$	0	1	2	3	4	5	-2	-1	0	1	2	3	4	5 (mod 8)
	Z_2	Z_2	Z_8	Z_2	O	Z_2	Z_{λ_i}	Z_2^2	Z_2^2	Z_2	Z_8	Z_2	O	Z_2 (i)

where λ_i is the 2-primary order of the image of the J-homomorphism. (That is, if $i + 1 \equiv 2^{\rho(i)} \bmod 2^{\rho(i)+1}$ then $\lambda_i = 2^{\rho(i)+1}$.) If $\lambda: Y \to P \wedge \text{Im } J$, then λ_* on the elements in $\Pi_* Y$ given by the image of J, the μ's of Barratt and Adams, the elements η_j and θ_j generate the image of λ_*.

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