 6/17/09

$X \in \pi_{2n}^s$; X represents a cobordism class of framed mflds M^{2n}. Say $\frac{1}{2}n > 4$

Try to simplify M by doing framed surgery as Ben talked about

More interesting case when n is odd. There is a quadratic form in $\pi_n M$..

$H^n(M,A) \longleftrightarrow \quad M \longrightarrow K(A,n)$

$$\underset{\text{(framed)}}{S^{2n}} \underset{\substack{\text{stable}\\\text{map}}}{\dashrightarrow} M \longrightarrow K(A,n)$$

(ie $S^{2n} \longrightarrow \Omega^\infty \Sigma^\infty_+ M \longrightarrow \Omega^\infty \Sigma^\infty K(A,n)$)

$n > 1$

Q : What is $\pi_{2n}(\underbrace{\Omega^\infty \Sigma^\infty K(A,n)}_{\text{free } E_\infty \text{ space on } K(A,n)})$?

$$\pi_{2n}\left(\bigvee_k (K(A,n)^{\wedge k})_{h\Sigma_k}\right) \cong \pi_{2n}\left(K(A,n) \vee (K(A,n) \wedge K(A,n))_{h\Sigma_2}\right)$$

thrown away things that are irrelevant

$$\simeq \pi_{2n}\left(K(A,n) \times (K(A,n)^{\wedge 2})_{h\Sigma_2}\right)$$

$$\simeq \cancel{\pi_{2n}(K(A,n))} \oplus \pi_{2n}(K(A,n)^{\wedge 2}{}_{h\Sigma_2})$$

first nonvanishing

$$\simeq (A \otimes A)_{\Sigma_2}$$

Action: depends on n : n even - just permutation

n odd - also a sign + permutation (graded commutative)

Upshot: M framed $2n$-mfld (ie $S^{2n} \underset{\text{stable map}}{\dashrightarrow} M$), then $\exists$ canonical map

$q : H^n(M;A) \longrightarrow (A \otimes A)_{\Sigma_2}$

Ex n even, $A = \mathbb{Z}$ $\quad H^n(M;\mathbb{Z}) \xrightarrow{q} \mathbb{Z}$

n odd $A = \mathbb{Z}, \mathbb{Z}/2$ $\quad H^n(M;A) \longrightarrow \mathbb{Z}/2$

Claim q is a quadratic map

(1) $q(0)=0$

(2) $q(x+y) = q(x) + q(y) + b(x,y)$

Pf (1) $M \longrightarrow K(A,n)$ constant map $\longmapsto q=0$.

(2) b is intersection form

$$b: H^n(M;A) \times H^n(M;A) \xrightarrow{\ \cup\ } H^{2n}(M;A) \longrightarrow A$$

$$x+y : M \xrightarrow{(x,y)} K(A,n)\times K(A,n) \xrightarrow[\text{natural addition}]{+} K(A,n)$$

$x,y : M \longrightarrow K(A,n)$

$$(x,y) \in H^n(M; A\oplus A)$$

$$q(x,y) \in \big((A\oplus A)\otimes(A\oplus A)\big)_{\mathbb{Z}_2} = (A\circledast A)_{\mathbb{Z}_2} \oplus (A\circledast A)_{\mathbb{Z}_2} \oplus (A\otimes A)$$

To justify (2), what we want to know:

$$q_{A\oplus A}(x,y) = (\underset{\checkmark}{q_A(x)}, \underset{\checkmark}{q_A(y)}, b(x,y))$$

follow $M \xrightarrow{(x,y)} K(A,n)\times K(A,n)$ by 2 projections.

More general statement $q_{A\oplus B}(x,y) = (q_A(x), q_B(y), b(x,y))$

$$b(x,y) \in H^{2n}(S^{2n}; A\otimes B) \cong A\otimes B$$

pectra:

$$\begin{tikzcd}
S^{2n} \arrow[r] & \Sigma_+^\infty M \arrow[r] \arrow[d] & \Sigma^\infty(K(A,n)\times K(B,n)) \arrow[d] \\
& \Sigma_+^\infty(M\times M) \arrow[ur, "{(x,y)}"] \arrow[r] \arrow[dr] & \Sigma^\infty(K(A,n)\wedge K(B,n)) \arrow[d] \\
& & \Sigma^\infty(K(A\otimes B, 2n))
\end{tikzcd}$$

Classified by $A\otimes B$, that is $b(x,y)$.

What can we say when $\pi = \mathbb{Z}/2$?

<u>Claim</u> This agrees with "Ben's form" (when defined).

What did Ben do?

$S^n \longrightarrow M \rightsquigarrow$ invariant $\{i$ behaves well.

in general position

We can look at the case $S^n \overset{i}{\hookrightarrow} M$ embedding

Let $N_{S/M}$ be the normal bundle of i, classified, w/ null homotopy to $BO(\infty)$ (framing of M)

$$\begin{tikzcd} S^n \arrow[r] \arrow[dr] & BO(n) \arrow[r] & BO(\infty) \\ & O(\infty)/O(n) \arrow[u] & \end{tikzcd}$$

What is $\pi_n(O(\infty)/O(n))$?

$$O(n+1)/O(n) \longrightarrow O(\infty)/O(n) \longrightarrow O(\infty)/O(n+1)$$ get LES in htpy

$O(n+1)/O(n) \cong S^n$, $O(\infty)/O(n+1)$ n-connected

$$\Rightarrow \pi_n(O(\infty)/O(n)) \longrightarrow \pi_n(O(\infty)/O(n+1)) = 0$$

$$\begin{tikzcd} & \pi_n(O(\infty)/O(n)) & \\ \pi_n(S^n) \simeq \mathbb{Z} \arrow[ur] & & \\ \pi_{n+1}(O(\infty)/O(n+1)) \arrow[u] & \pi_{n+1}(S^{n+1}) \cong \mathbb{Z} \arrow[l, two heads] \arrow[ul, "\cdot 1+(-1)^{n+1}"'] & \end{tikzcd}$$

$$\Rightarrow \pi_n(O(\infty)/O(n)) = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}/2 & \text{odd} \end{cases}$$

From now on assume n odd.

Suppose $x \in H^n(M; \mathbb{Z}/2)$ Poincaré dual to $i: S^n \hookrightarrow M$.

What is $q(x)$?

$$S^{2n} \longrightarrow \Omega^\infty \Sigma^\infty_+ M \longrightarrow \Omega^\infty\Sigma^\infty(M/M-i(S^n)) \longrightarrow \Omega^\infty\Sigma^\infty K(\mathbb{Z}/2, n)$$

$$\Omega^\infty\Sigma^\infty(M/M-i(S^n)) \cong \Omega^\infty\Sigma^\infty(N_{S/M}/N_{S/M} - \text{zero section})$$

Depends only on the invariant (bundle w/ stable trivialization)

- Suppose $N_{S^n/M}$ is trivial (compatible w/ framing of M) = Ben's invariant = 0.

Spectra:
$$\begin{tikzcd}
S^{2n} \arrow[r] \arrow[dr, "\text{inclusion}"'] & \Sigma^\infty \left(S^n\times D^n / S^n \times \partial D^n \right) \arrow[r] & \Sigma^\infty D^n/S^{n-1} \simeq S^n \arrow[r] & K(\mathbb{Z}/2, n) \\
& S^{2n} \vee S^n \arrow[ur, "\text{projection}"'] & &
\end{tikzcd}$$
($\partial D^n = S^{n-1}$; $\Sigma^\infty(S^n\times D^n/S^n\times\partial D^n) \cong S^{2n}\vee S^n$)

composition = 0

- Suppose $N_{S^n/M}$ not trivial (compatible w/ framing) = Ben's invariant = 1.

Need an example of $q \equiv 0$. $\rightsquigarrow S^n \times S^n$

What is $q(x)$? Given by Ben's formula (reality check)

A quadratic space over k is

Def Let k be a field (in our case $k = \mathbb{F}_2$). a k-vector space V, $q: V \to k$ s.t

$$q(\lambda x) = \lambda^2 q(x)$$
$$q(x+y) = q(x) + q(y) + b(x,y) \quad \text{where } b(x,y) \text{ nondeg bilinear form.}$$

Def A quadratic space $q: V \to k$ is trivial if $\dim V = 2m$, and $\exists$ subspace $W \subset V$ s.t $\dim W = m$, $q|_W \equiv 0$.

$$\left(\quad 0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0 \quad \right)$$
$$V/W \cong W^* \quad (q|_W \equiv 0)$$

Def Witt group $\mathrm{Witt}(k) = \mathrm{Gr}\left(\text{iso classes of quadratic spaces}/k, \oplus \right) / \text{trivial quadratic spaces}$

$(V, q) \oplus (V, -q)$ is a trivial quadratic space

$V \xrightarrow{\text{diag}} V \oplus V$, q vanishes.

(so didn't even have to go complete)

Claim: Today's lecture produces a map

$$\text{Kervaire}: \pi_{2n}^{s} \longrightarrow \mathrm{Witt}(\mathbb{F}_2)$$
$$M^{2n} \longmapsto (H^{n}(M;\mathbb{F}_2), q)$$

Additive

Claim If M is boundary of N where N is a framed $(2n+1)$-mfld then $\mathrm{Kervaire}(M) = 0$, ie

$(H^n(M;\mathbb{F}_2), q)$ is a trivial quadratic space.

$$\cdots \to H^n(N;\mathbb{F}_2) \xrightarrow{\alpha} H^n(\partial N;\mathbb{F}_2) \xrightarrow{\beta} H^{n+1}(N,\partial N;\mathbb{F}_2) \to \cdots$$

$$\begin{array}{c} H^n(\partial N;\mathbb{F}_2) \\ \wr\wr \end{array}$$

$$0 \to \mathrm{Im}(\alpha) \to H^n(M;\mathbb{F}_2) \to \mathrm{Im}(\beta) \to 0$$

Intersection pairing gives $\mathrm{Im}(\alpha) \cong \mathrm{Im}(\beta)$

$\mathrm{Im}(\alpha) \oplus \mathrm{Im}(\beta) \cong H^n(M)$.

Need: q vanishes on $\mathrm{Im}(\alpha)$ $\Longrightarrow$

$x \in \mathrm{Im}(\alpha) \subseteq H^n(M;\mathbb{F}_2)$

$$\begin{tikzcd} M \arrow[r, "x"] \arrow[d, hook] & K(\mathbb{F}_2, n) \\ N \arrow[ur, dashed] & \end{tikzcd}$$

$M = \partial M$

$$\begin{tikzcd} S^{2n} \arrow[r] \arrow[d] & \Sigma^\infty_+ M \arrow[r] \arrow[d] & \Sigma^\infty K(\mathbb{F}_2, n) \\ D^{2n+1} \arrow[r] & \Sigma^\infty_+ N \arrow[ur] & \end{tikzcd}$$

D^{2n+1}
$\parallel$
0

so $q(x) = 0$.

$\square$

$, \; = \partial N$

$\mathrm{Witt}(\mathbb{F}_2) = \mathbb{Z}/2$

We won't show this, but will produce $\mathrm{Witt}(\mathbb{F}_2) \xrightarrow{\mathrm{Arf}} \mathbb{Z}/2$

<u>Construction</u>: If (V,q) is a quadratic space $/k$ $\rightsquigarrow$

$\mathrm{Cliff}(V,q) = $ free alg generated by V / relations : $v^2 = q(v)$

$\mathrm{Cliff}(V,q)$ is $\mathbb{Z}/2$-graded, every $v \in V$ has odd deg.

$\dim \mathrm{Cliff}(V,q) = 2^{\dim V}$

If V has a basis $v_1, \dots, v_n$, $\mathrm{Cliff}(V,q)$ has a basis $v_{i_0} \cdots v_{i_k}$. $i_0 < i_1 < \dots < i_k$

$\dim \mathrm{Cliff}(V,q)_0 = 2^{\dim V - 1}$

Say V has even dimension $2m$.
$\dim \mathrm{Cliff}(V,q)_0 = 2^{2m-1}$ (can't be a matrix algebra, not a central simple algebra $/k$)

It has interesting center.

$\mathrm{Center}(\mathrm{Cliff}(V,q)_0)$ Commutative étale k-algebra of rank 2 (semisimple)

Either $k \times k$ or some quadratic extension.

$$\mathrm{Arf}(V,q) = \begin{cases} 0 & \text{if } \mathrm{Center}(\mathrm{Cliff}(V,q)_0) \simeq \mathbb{F}_2 \times \mathbb{F}_2 \\ 1 & \text{if } \mathrm{Center}(\mathrm{Cliff}(V,q)_0) = \mathbb{F}_4 \end{cases}$$

$\mathrm{Arf}(V,q) \in H^1_{ét}(k, \mathbb{Z}/2)$
étale 2-fold covers.

(For general k: Discriminant: $\mathrm{Witt}(k)_0 \longrightarrow H^1_{ét}(k; \mathbb{Z}/2)$)

<u>Claim</u> If $k = \mathbb{F}_2$, $\mathrm{Arf} : \mathrm{Witt}(\mathbb{F}_2) \longrightarrow \mathbb{Z}/2$ is iso.

(Can prove this doing surgery on (V,q), if $\exists\, v \in V$ s.t. $q(v) = 0$, can change (V,q) by something of smaller dimension)

$\rightsquigarrow V = \mathbb{F}_2 x \oplus \mathbb{F}_2 y$ $\quad q(x) = q(y) = q(x+y) = 1$

<u>Ex</u> $M =$ torus of dim 2, framing is left invariant