

$X \in \pi_{2n}^S$ ;  $X$  represents a cobordism class of framed mflds  $M^{2n}$

Try to simplify  $M$  by doing framed surgery as Ben talked about

More interesting case when  $n$  is odd. There is a quadratic form in  $\pi_n M$ .

$$H^n(M, A) \leftarrow M \longrightarrow K(A, n)$$

(framed)  $S^{2n} \xrightarrow{\text{stable map}} M \longrightarrow K(A, n)$   
 (ie  $S^{2n} \rightarrow \Omega^\infty \Sigma_+^\infty M \longrightarrow \Omega^\infty \Sigma^\infty K(A, n)$ )

$n > 1$

Q: What is  $\pi_{2n}(\Omega^\infty \Sigma^\infty K(A, n))$ ?  
 free  $E_\infty$ -space on  $K(A, n)$

$$\pi_{2n} \left( \bigvee_k (K(A, n)^{\wedge k})_{n\Sigma_k} \right) \cong \pi_{2n} \left( K(A, n) \vee (K(A, n) \wedge K(A, n))_{n\Sigma_2} \right)$$

thrown things away that are irrelevant

$$\cong \pi_{2n} \left( K(A, n) \times (K(A, n))^{\wedge 2}_{n\Sigma_2} \right)$$

$$\cong \pi_{2n} \left( \cancel{K(A, n)} \right) \oplus \pi_{2n} \left( K(A, n)^{\wedge 2}_{n\Sigma_2} \right)$$

first nonvanishing

$$\cong (A \otimes A)_{\Sigma_2}$$

Action: depends on  $n$ :  $n$  even - just permutation  
 $n$  odd - also a sign + permutation (graded commutative)

Upshot:  $M$  framed  $2n$ -mfd (ie  $S^{2n} \xrightarrow{\text{stable map}} M$ ), then  $\exists$  canonical map

$$q: H^n(M; A) \longrightarrow (A \otimes A)_{\Sigma_2}$$

Ex  $n$  even,  $A = \mathbb{Z}$   $H^n(M; \mathbb{Z}) \xrightarrow{q} \mathbb{Z}$

$n$  odd  $A = \mathbb{Z}, \mathbb{Z}/2$   $H^n(M; A) \longrightarrow \mathbb{Z}/2$

Claim  $q$  is a quadratic map

(1)  $q(0) = 0$

(2)  $q(x+y) = q(x) + q(y) + b(x,y)$

pf (1)  $M \rightarrow K(A,n)$  constant map  $\mapsto q=0$ .

(2)  $b$  is intersection form

$$b: H^n(M; A) \times H^n(M; A) \xrightarrow{u} H^{2n}(M; A) \rightarrow A$$

$$x, y: M \rightarrow K(A, n) \quad x+y: M \xrightarrow{(x,y)} K(A, n) \times K(A, n) \xrightarrow{\substack{+ \\ \text{natural} \\ \text{addition}}} K(A, n)$$

$$(x, y) \in H^n(M; A \oplus A)$$

$$q(x, y) \in ((A \oplus A) \otimes (A \oplus A))_{\Sigma_2} = (A \otimes A)_{\Sigma_2} \oplus (A \otimes A)_{\Sigma_2} \oplus (A \otimes A)$$

To justify (2), what we want to know:

$$q_{A \oplus A}(x, y) = (q_A(x), q_A(y), b(x, y))$$

follow  $M \xrightarrow{(x,y)} K(A, n) \times K(A, n)$  by 2 projections.

More general statement  $q_{A \otimes B}(x, y) = (q_A(x), q_B(y), b(x, y))$   
 $H^{2n}(S^{2n}; A \otimes B)$   
 $A \otimes B$

spectra:  $S^{2n} \rightarrow \Sigma_+^{\infty} M \rightarrow \Sigma^{\infty} (K(A, n) \times K(B, n))$   
 $\downarrow$   
 $\Sigma_+^{\infty} (M \times M) \xrightarrow{(x,y)} \Sigma^{\infty} (K(A, n) \wedge K(B, n))$   
 $\downarrow$   
 $\Sigma^{\infty} (K(A \otimes B, 2n))$

Classified by  $A \otimes B$ , that is  $b(x, y)$ .

What can we say when  $\pi = 4/2$ ?

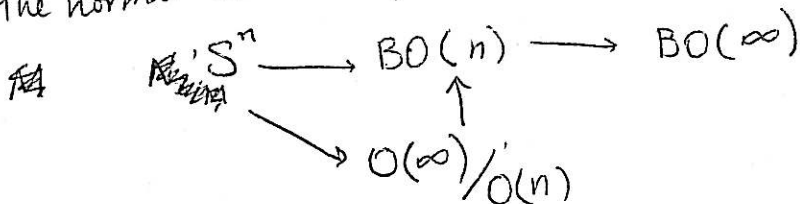
Claim This agrees with "Ben's form" (when defined).

What did Bendo?

$S^n \rightarrow M \rightsquigarrow$  invariant  $\begin{cases} 0 \\ 1 \end{cases}$  behaves well.  
in general position

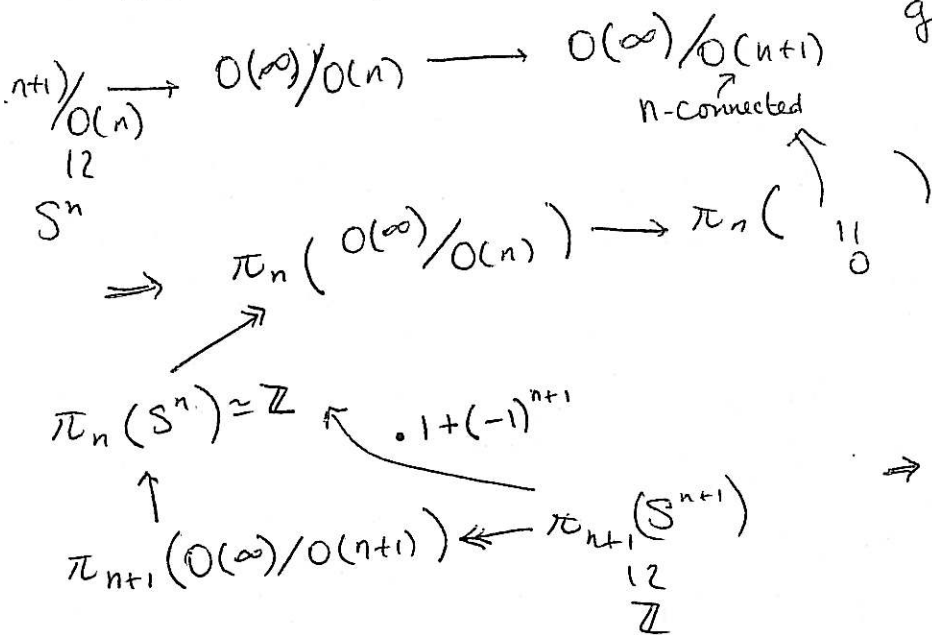
We can look at the case  $S^n \xrightarrow{i} M$  embedding

Let  $N_{S/M}$  be the normal bundle of  $i$ , classified, w/ null homotopy to  $BO(\infty)$  (framing of  $M$ )



What is  $\pi_n(O(\infty)/O(n))$ ?

get LES in htpy



$$\pi_n(O(\infty)/O(n)) = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}/2 & n \text{ odd} \end{cases}$$

From now on assume  $n$  odd.

Suppose  $x \in H^n(M; \mathbb{Z}/2)$  Poincaré dual to  $i: S^n \hookrightarrow M$ .

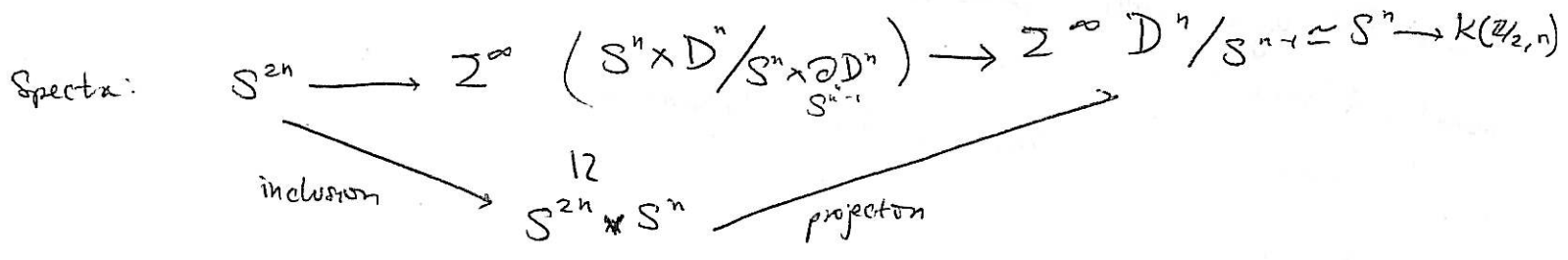
What is  $g(x)$ ?

$$S^{2n} \longrightarrow \Omega^\infty \Sigma_+^\infty M \longrightarrow \Omega^\infty \Sigma_+^\infty (M/M-i(S^n)) \longrightarrow \Omega^\infty \Sigma K(\mathbb{Z}/2, n)$$

$$\Omega^\infty \Sigma^\infty (N_{S/M}/N_{S/M} - \text{zero section})$$

Depends only on the invariant (bundle w/ stable trivialization)

Suppose  $N_{S^n/M}$  is trivial (compatible w/ framing of  $M$ ) = Ben's invariant = 0.



composition = 0

Suppose  $N_{S^n/M}$  not trivial (compatible w/ framing) = Ben's invariant = 1.

Need an example of  $q \equiv 0$ .  $\leadsto S^n \times S^n$

What is  $q(x)$ ? Given by Ben's formula (reality check)

Def Let  $k$  be a field (in our case  $k = \mathbb{F}_2$ ). A quadratic space over  $k$  is a  $k$ -vector space  $V$ ,  $q: V \rightarrow k$  s.t.

$$q(\lambda x) = \lambda^2 q(x)$$

$$q(x+y) = q(x) + q(y) + b(x,y)$$

where  $b(x,y)$  nondeg bilinear form.

Def A quadratic space  $q: V \rightarrow k$  is trivial if  $\dim V = 2m$ , and  $\exists$  subspace  $W = V$  s.t.  $\dim W = m$ ,  $q|_W \equiv 0$ .

$$\left( 0 \rightarrow W \rightarrow V \rightarrow \begin{matrix} V/W \\ \cong W^* \\ (q|_W \equiv 0) \end{matrix} \rightarrow 0 \right)$$

Def Witt group  $Witt(k) = \text{Gen}(\text{iso classes of quadratic spaces } / k, \oplus)$  / trivial quadratic spaces

$(V, q) \oplus (V, -q)$  is a trivial quadratic space

$$V \xrightarrow{\text{diag}} V \oplus V, q \text{ vanishes.}$$

(So didn't even have to go complete)

Claim: Today's lecture produces a map

$$\text{Kervaire} : \begin{array}{ccc} \pi_{2n}^s & \longrightarrow & \text{Witt}(\mathbb{F}_2) \\ M^{2n} & \longmapsto & (H^n(M; \mathbb{F}_2), q) \end{array}$$

Additive

Claim If  $M$  is boundary of  $N$  where  $N$  is a framed  $(2n+1)$ -manifold then  $\text{Kervaire}(M) = 0$ , i.e.

$(H^n(M; \mathbb{F}_2), q)$  is a trivial quadratic space.

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^n(N; \mathbb{F}_2) & \xrightarrow{\alpha} & H^n(\partial N; \mathbb{F}_2) & \xrightarrow{\beta} & H^{n+1}(N, \partial N; \mathbb{F}_2) \rightarrow \cdots \\ & & & & \downarrow \cong & & \\ 0 & \rightarrow & \text{Im}(\alpha) & \rightarrow & H^n(M; \mathbb{F}_2) & \rightarrow & \text{Im}(\beta) \rightarrow 0 \end{array}$$

Intersection pairing gives  $\text{Im}(\alpha) \cong \text{Im}(\beta)$   
 $\text{Im}(\alpha) \oplus \text{Im}(\beta) \cong H^n(M)$

Need:  $q$  vanishes on  $\text{Im}(\alpha)$

$$x \in \text{Im}(\alpha) \in H^n(M; \mathbb{F}_2) \quad \Rightarrow$$

$$\begin{array}{ccc} x: M & \longrightarrow & K(\mathbb{F}_2, n) \\ \downarrow & \nearrow & \\ N & & \end{array}$$

$$M = \partial N$$

$$S^{2n} \longrightarrow \Sigma_+^\infty M \longrightarrow \Sigma^\infty K(\mathbb{F}_2, n)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ D^{2n+1} & \longrightarrow & \Sigma_+^\infty N \\ \cong & & \nearrow \end{array}$$

$$\text{so } q(x) = 0.$$

□

$$\text{Witt}(\mathbb{F}_2) = \mathbb{Z}/2$$

We won't show this, but will produce  $\text{Witt}(\mathbb{F}_2) \xrightarrow{\text{Arf}} \mathbb{Z}/2$

Construction: If  $(V, q)$  is a quadratic space /  $k \rightarrow$

$\text{Cliff}(V, q) =$  free alg generated by  $V$  / relations:  $v^2 = q(v)$

$\text{Cliff}(V, q)$  is  $\mathbb{Z}/2$ -graded, every  $v \in V$  has odd deg.

$$\dim \text{Cliff}(V, q)_0 = 2^{\dim V}$$

$\hookrightarrow$  If  $V$  has a basis  $v_1, \dots, v_n$ ,  $\text{Cliff}(V, q)$  has a basis  $v_{i_0} \dots v_{i_k}$ ,  $i_0 < i_1 < \dots < i_k$

$$\dim \text{Cliff}(V, q)_0 = 2^{\dim V - 1}$$

Say  $V$  has even dimension  $2m$ .

$$\dim \text{Cliff}(V, q)_0 = 2^{2m-1}$$

(can't be a matrix algebra, not a central simple algebra /  $k$ )

It has interesting center.

Center  $(\text{Cliff}(V, q)_0)$

Commutative <sup>étale</sup>  $k$ -algebra of rank 2 (semisimple)

Either  $k \times k$  or some quadratic extension.

$$\text{Arf}(V, q) = \begin{cases} 0 \\ 1 \end{cases}$$

if center  $(\text{Cliff}(V, q)_0) \cong \mathbb{F}_2 \times \mathbb{F}_2$

if center  $(\text{Cliff}(V, q)_0) = \mathbb{F}_4$

$$\text{Arf}(V, q) \in H_{\text{ét}}^1(k, \mathbb{Z}/2)$$

étale 2-fold covers.

(For general  $k$ : Discriminant:  $\text{Witt}(k)_0 \rightarrow H_{\text{ét}}^1(k, \mathbb{Z}/2)$ )

Claim If  $k = \mathbb{F}_2$ ,  $\text{Arf}: \text{Witt}(\mathbb{F}_2) \rightarrow \mathbb{Z}/2$  is iso.

(Can prove this doing surgery on  $(V, q)$ , if  $\exists v \in V$  s.t.  $q(v) = 0$ , can change  $(V, q)$  by something of smaller dimension)

$$\rightarrow V = \mathbb{F}_2 \times \oplus \mathbb{F}_2 y \quad q(x) = q(y) = q(x+y) = 1$$

Ex  $M =$  torus of dim 2, framing is left invariant