Summer Seminar - Denis Cisinski - Homotopy Weaking of Algebras 7/22/09

Part I

$\boxed{T_1 \circlearrowright \text{Graphs (directed)} \quad \text{monad}} \underset{\text{forget}}{\overset{\text{free}}{\rightleftarrows}}$ Cat $= T_1$-algebras

(in this example $\mathcal{C} = \begin{smallmatrix} 1 \\ \uparrow\uparrow \\ 0 \end{smallmatrix}$)

$\rightsquigarrow$

Δ category of simplices

$\rightsquigarrow$

sSets + Joyal model structure for quasi-categories

$\mathcal{C}$ small category, $\hat{\mathcal{C}} = \mathrm{Func}(\mathcal{C}^{op}, \text{sets})$ (presheaves)

$\hat{\mathcal{C}} \circlearrowright T$ monad

nice monad i e, $1 \xrightarrow{\eta} T \quad T \circ T \xrightarrow{\mu} T$ give pullback squares for maps

1) T Cartesian

2) T familially representable (Lenster)
parametric right adjoints monads (Weber)

$$\hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}/T(*)$$
$$X \longmapsto \begin{matrix} T(X) \\ \downarrow \\ T(*) \end{matrix}$$
preserves limits.

Examples

1) $C = \mathbb{G}_1 = \{0 \rightrightarrows 1\}$ $\quad \hat{\mathbb{G}}_1 =$ graphs

$T = T_1 =$ free cat monad

2) $C = \mathbb{G}_n$ $\quad \hat{\mathbb{G}}_n = n$-graphs

$$0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} 2 \rightrightarrows \cdots n \qquad \begin{matrix} ss = ts \\ tt = st \end{matrix}$$

$T = T_n$ free (strict) n-cat monad $\quad 1 \leq n \leq \infty$

3) $C = \mathbb{M}$ $\quad \hat{\mathbb{M}} =$ multigraphs

$T = T_{mult} =$ free $\left\{ \begin{matrix} \text{multicategory monad} \\ \text{(colored operad)} \end{matrix} \right.$

$\mathbb{M} \quad \begin{pmatrix} (n,1) \\ \uparrow n+1 \text{ maps} \\ 0 \end{pmatrix} \quad n \geq 0$

$(X(x_1, \dots, x_n; x))$

$Kl(T)$ = Kleisli category of T = free T-algebras
$\cap$
T-alg
has a factorization system {free maps, generic maps}

generic = unique right lifting property wrt free maps.

$\Theta_0 \subset \hat{C}$ full subcat $A \in \Theta_0$ if $\exists$ generic map $T(C) \longrightarrow T(A)$ w/ $C \in C$.

Θ_T full subcat of T-alg spanned by $T(A)$, $A \in \Theta_0$.

ie 1) $\Theta_{T_1} \cong \Delta$

2) $\Theta_n := \Theta_{T_n}$, Θ_n^{op} = Joyal category of n-discs.

3) $\Theta_{T_{mult}} = \Omega$ Moerdijk - Weiss

$\Theta_T \hookrightarrow T\text{-Alg}$ $\qquad N(X)(A) = Hom_{T\text{-Alg}}(TA, X)$

$\hat{\Theta}_T \underset{N}{\overset{\tau}{\rightleftarrows}} T\text{-Alg}$

<u>Thm</u> (Weber) $N: T\text{-Alg} \longrightarrow \hat{\Theta}_T$ is fully faithful and its essential image consists of $X \in \hat{\Theta}_T$ satisfying the Segal condition.

Segal conditions:
$A \in \Theta_0 \subset \hat{C}$

$$\begin{array}{c} c \\ \downarrow \\ A \end{array} \qquad \begin{tikzcd} C/A \arrow[r] \arrow[rrr, bend right, "Seg(A)"'] & C \ \Theta_0 \arrow[r, "T"] & \Theta_T \arrow[r, hook] & \hat{\Theta}_T \end{tikzcd}$$

Segal condition for $X \in \hat{\Theta}_T$: $\forall A \in \Theta_0$

$$X(A) \cong \varprojlim_{(C/A)^{op}} Hom_{\hat{\Theta}_T}(Seg(A), X).$$

Homotopy theories in presheaves categories

A small category (in examples: $A = \Theta_T$, $A = \Theta_T \times \Delta$)

Def $W \subset \mathrm{Arr}\,\hat{A}$. W is a localizer if

1) W has 2 out of 3 property

2) $\{\text{right lifting property wrt mono}\} \subset W$

3) $W \cap \{\text{mono}\}$ is stable by pushouts and transfinite compositions.

S class of maps, define $W(S) = \bigcap_{S \subset W \text{ localizer}} W$ (smallest localizer containing S)

A localizer is accessible if $W = W(S)$ for S a small set of maps.

Prop W accessible localizer $\Leftrightarrow$ $\hat{A}$ has a model structure cofibrantly generated with Cof = monos, weak-equiv = W

hocolim is defined wrt minimal localizer.

Ex $A = \Delta$, $S = \{\Delta_n \longrightarrow \Delta_0 \mid n \geq 0\}$

$W(S)$ is the usual class of weak equivalences.

Def A localizer W is regular if $\forall\, X \in \hat{A}$

$$A/X \xrightarrow{\varphi_X} \hat{A}$$

$$\operatorname*{hocolim}_{A/X} \varphi_X \longrightarrow X \in W$$

Prop The minimal regular localizer ~~theory~~ is accessible.

Prop Any Θ_n-localizer is regular.

Joyal model structure:

$\hat{C} \circlearrowleft T \rightsquigarrow \Theta_T$

The Joyal localizer (associated to T) $W_J = W_{J,T}$ is the regular localizer generated by maps of shape:

$$\operatorname*{hocolim}_{C/A} Seg(A) \longrightarrow T(A) \qquad A \in \Theta_0$$

More on regular localizers

A small cat

$\widehat{A \times \Delta}$ $\qquad X \times \Delta_1 \longrightarrow X$ not enough.

__Thm__ If we take the regular localizer generated by $X \times \Delta_1 \longrightarrow X$, we get the class of termwise simplicial weak equivalences.

(have to impose regularity).

Consider a localizer W on $\hat{A}$.

W_Δ = localizer on $\widehat{A \times \Delta}$ (simplicial presheaves) generated by termwise W-equivalences and by $X \times \Delta_1 \longrightarrow X$.

Get Quillen equiv: $(\hat{A}, W) \simeq (\widehat{A \times \Delta}, W_\Delta)$

__Prop__ W accessible (regular) then so is W_Δ.

__Cor__ If W is a regular localizer on $\hat{A}$, then W_Δ is the localizer on W_Δ generated by • termwise simplicial weak equivalences

• termwise W-equivalences.

__Rezk model structure__

$\hat{C} \circlearrowleft T \qquad \Theta_T$

$W_R = W_{R,T}$ ~~is the regular localizer on $\widehat{\Delta \times \Theta_T}$~~ (on $\widehat{\Theta_T \times \Delta}$)

$= (W_{J,T})_\Delta$

$$\begin{tikzcd} + \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & I \end{tikzcd}$$ disjoint global sections

Another description of W_R: pick an interval I in $\hat{\Theta}_T$

$X \times I \longrightarrow X \in W_J$.

W_R is the regular localizer generated by $\cdot\ X \times \Delta_1 \longrightarrow X$

$\cdot\ X \times I \longrightarrow X$

$\cdot\ \forall A \in \Theta_0 \qquad \operatorname*{hocolim}_{C/A} \mathrm{Seg}(A) \longrightarrow T(A)$

$$(\hat{\Theta}_T, W_J) \underset{Q}{\sim} (\widehat{\Theta_T \times \Delta_1}, W_R)$$

Iterative procedure for n-categories

<u>Part II</u>

<u>Wreath products</u>

$$\begin{array}{c} A \\ \downarrow \\ \Gamma \end{array} \quad = \{\text{finite ptd sets}\}^{op} \quad \rightsquigarrow \quad A \wr C \quad \text{for any other } C$$

Ex $\cdot\ \mathbb{G}_1 \longrightarrow \Gamma$

$\mathbb{G}_1 \wr A$ full subcategory of the category of graphs enriched in $\hat{A}$.

for $X \in$ representable presheaf.

$\Delta_0 = \{X\} \qquad \Delta_1(X) = 0 \xrightarrow[X]{} 1$

$\widehat{\mathbb{G}_1 \wr A} = \mathrm{graph}(\hat{A})$

$= \hat{A}$-enriched graphs

$\hat{C} \circlearrowleft T \qquad \widehat{\mathbb{G}_1 \wr C} \circlearrowleft T_1 \wr T \quad$ s.t $\quad T_1 \wr T\text{-Alg} =$ categories enriched in T-algebras

$= \mathrm{Cat}(T\text{-Alg})$.

$\cdot\ \Delta \wr A$ full subcat of $\mathrm{Cat}(\hat{A})$ spanned by objects of shape

$\Delta_0 \qquad \Delta_n(a_1, \dots, a_n) \qquad 0 \xrightarrow{a_1} 1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} n$

free category generated by

$\mathrm{Hom}(0,2) = a_1 \times a_2 \qquad$ (as presheaves)

$\mathrm{Hom}(0,k) = a_1 \times \cdots \times a_k$

$\hat{A} \circlearrowleft \mathrm{id}$

$T_1 \int id$

$\Theta_{T_1 \int id_{\hat{A}}} = \Delta \int A$

$N : \mathrm{Cat}(\hat{A}) \longrightarrow \widehat{\Delta \int A}$

<u>Thm</u> (C. Berger) $\Delta \int \Theta_n = \Theta_{n+1}$

A small category. We have a Joyal localizer $W_{J, T_1 \int id_{\hat{A}}}$ on $\hat{A}$.

Consider an accessible localizer on $\hat{A}$. Assume W is stable by finite products

The regular localizer generated by

$$W_{J, T_1 \int id_{\hat{A}}}$$

and by

$$\Delta_1(X) \to \Delta_1(Y) \quad \text{for } X \to Y \text{ in } W$$

defines the homotopy theory of $(\infty, 1)$-categories enriched in (fibrant) presheaves on A

Apply this for $A = \Theta_n$, $W = W_{J, T_n}$, this defines a model category structure on $\hat{\Theta}_{n+1}$

<u>Thm</u> This structure is the same as the one associated to $W_{J, T_{n+1}}$

$$\hat{A} \longrightarrow \Delta_0 \amalg \Delta_0 \backslash \widehat{\Delta \int A}$$
$$X \longmapsto 0 \xrightarrow[X]{} 1$$

is a left Quillen functor and is right adjoint to the functor

$$(X, a, b) \longmapsto X(a, b)$$

<u>Groupoid-like algebras</u>

Localization

$$\begin{array}{ccc} S & \subset & C \\ \downarrow & & \downarrow \\ S^{-1}S & \xrightarrow{J} & S^{-1}C \end{array}$$

enough to know how to localize $S^{-1}S$.

Groupoids $\longleftrightarrow$ Cat has left adj, right adj (localize / maximal sub).

Assume that we have the following assumptions:

1) $A \in \Theta_0$, $|NC/A| \simeq *$

2 Θ_T is a local test category.

A is a local test category if $\exists$ I interval s t $\forall$ $a \in A$

$$|A/(a \times I)| \sim pt$$

$\Leftrightarrow$ $\hat{A}$ has a model structure $\sim$ $Top/|A|$

All the examples given satisfy these conditions: Θ_n, Ω

W_K is a regular localizer of $\hat{\Theta}_T$ generated by maps between representable presheaves

Prop $W_{J,T} \subset W_K$

$wT\text{-}Alg = DK(\hat{\Theta}_T, W_J)$

weak T-algebras (Dwyer-Kan localization)

$grwT\text{-}Alg = DK(\hat{\Theta}_T, W_K) \sim$ spaces $/B\Theta_T$

groupoid-like

$$grwT\text{-}Alg \underset{i}{\overset{\text{left-adjoint}}{\leftrightarrows}} wT\text{-}Alg$$

("maximal subgroupoid-like")

Prop the functor i also has a right adjoint