



$Kl(T) =$  Kleisli category of  $T =$  free  $T$ -algebras

$\uparrow$   
 $T$ -alg

has a factorization system  $\{ \text{free maps, generic maps} \}$

generic = unique right lifting property wrt free maps.

$\mathbb{H}_0 \subset \hat{C}$  full subcat.  $A \in \mathbb{H}_0$  if  $\exists$  generic map  $T(C) \rightarrow T(A)$   
 $w/ C \in C.$

$\mathbb{H}_T$  full subcat of  $T$ -alg spanned by  $T(A), A \in \mathbb{H}_0.$

ie  $\mathbb{H}_{T_1} \cong \Delta$

2)  $\mathbb{H}_n := \mathbb{H}_{T_n}$ ,  $\mathbb{H}_n^{op} =$  Joyal category of  $n$ -discs.

3)  $\mathbb{H}_{T_{mult}} = \Omega$  Moerdijk-Weiss

$\mathbb{H}_T \rightarrow T\text{-Alg}$

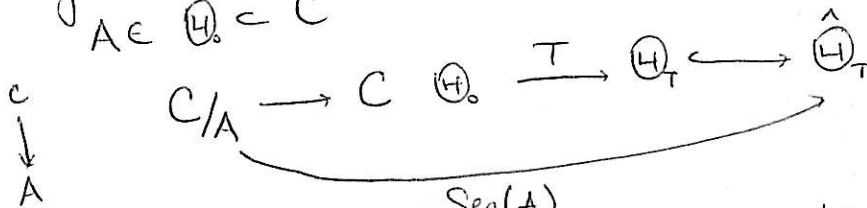
$$N(X)(A) = \text{Hom}_{T\text{-Alg}}(TA, X)$$

$\hat{\mathbb{H}}_T \xrightleftharpoons[N]{\tau} T\text{-Alg}$

Thm (Weber)  $N: T\text{-Alg} \rightarrow \hat{\mathbb{H}}_T$  is fully faithful and its essential image consists of  $X \in \hat{\mathbb{H}}_T$  satisfying the Segal condition.

Segal conditions:

$A \in \mathbb{H}_0 \subset \hat{C}$



Segal condition for  $X \in \hat{\mathbb{H}}_T$ :  $\forall A \in \mathbb{H}_0$

$$X(A) \cong \varinjlim_{(C/A)^{op}} \text{Hom}_{\hat{\mathbb{H}}_T}(\text{Seg}(A), X)$$

# Homotopy theories in presheaves categories

A small category (in examples  $\mathcal{A} = \mathbb{H}_T, \mathcal{A} = \mathbb{H}_T \times \Delta$ )

Def  $W \subset \text{Arr } \hat{\mathcal{A}}$ .  $W$  is a localizer if

- 1)  $W$  has 2 out of 3 property
- 2) {right lifting property wrt mono}

$\wedge$   
 $W$

3)  $W \cap \{\text{mono}\}$  is stable by pushouts and transfinite compositions.

$S$  class of maps, define  $W(S) = \bigcap_{S \subset W \text{ localizer}} W$

(smallest localizer containing  $S$ )

A localizer is accessible if  $W = W(S)$  for  $S$  a small set of maps.

Prop  $W$  accessible localizer  $\iff \hat{\mathcal{A}}$  has a model structure cofibrantly generated

with  $\text{Cof} = \text{monos}$   
 $\text{Weak-equiv} = W$

Ex  $\mathcal{A} = \Delta, S = \{ \Delta_n \rightarrow \Delta_0 \mid n \geq 0 \}$

$W(S)$  is the usual class of weak equivalences.

Def A localizer  $W$  is regular if  $\forall X \in \hat{\mathcal{A}}$

$$A/X \xrightarrow{\mathcal{U}_X} \hat{\mathcal{A}}$$

$$\text{hocolim}_{A/X} \mathcal{U}_X \rightarrow X \in W$$

Prop The minimal regular localizer  $W_{\text{reg}}$  is accessible.

Prop Any  $\mathbb{H}_n$ -localizer is regular.

Joyal model structure:

$$\hat{\mathcal{C}} \mathcal{J} T \rightsquigarrow \mathbb{H}_T$$

The Joyal localizer (associated to  $T$ )  $W_J = W_{J,T}$  is the regular localizer generated by maps of shape:

hocolim is defined wrt minimal localizer.

$$\text{hocolim Seg}(A) \xrightarrow{c/A} T(A)$$

$$A \in \mathbb{H}_0$$

More on regular localizers:

$$A \xrightarrow{\text{small cat}} \hat{A} \times \Delta$$

$$X \times \Delta_1 \longrightarrow X \quad \text{not enough.}$$

Then If we take the regular localizer generated by  $X \times \Delta_1 \longrightarrow X$ , we get the class of termwise simplicial weak equivalences.

(have to impose regularity).

Consider a localizer  $W$  on  $\hat{A}$ .

$W_\Delta =$  localizer on  $\hat{A} \times \Delta$  (simplicial presheaves) generated by termwise  $W$ -equivalences and by  $X \times \Delta_1 \longrightarrow X$ .

$$\text{Get Quillen equiv: } (\hat{A}, W) \simeq (\hat{A} \times \Delta, W_\Delta)$$

Prop  $W$  accessible then so is  $W_\Delta$ .  
(regular)

Cor If  $W$  is a regular localizer on  $\hat{A}$ , then  $W_\Delta$  is the localizer on  $W_\Delta$  generated by termwise simplicial weak equivalences and termwise  $W$ -equivalences.

Rezk model structure

$$\hat{C} \hookrightarrow T \quad \mathbb{H}_T$$

$$W_R = W_{R,T} \text{ is the regular localizer on } \hat{\Delta} \times \mathbb{H}_T \text{ (on } \mathbb{H}_T \times \Delta \text{)}$$

$$= (W_{J,T})_\Delta$$

$$\begin{array}{ccc} + & - & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & I \end{array} \quad \text{disjoint global sections}$$

Another description of  $W_R$ : pick an interval  $I$  in  $\mathbb{H}_T$

$$X \times I \longrightarrow X \in W_J$$

$W_R$  is the regular localizer generated by

- $\cdot X \times \Delta_1 \rightarrow X$
- $\cdot X \times I \rightarrow X$

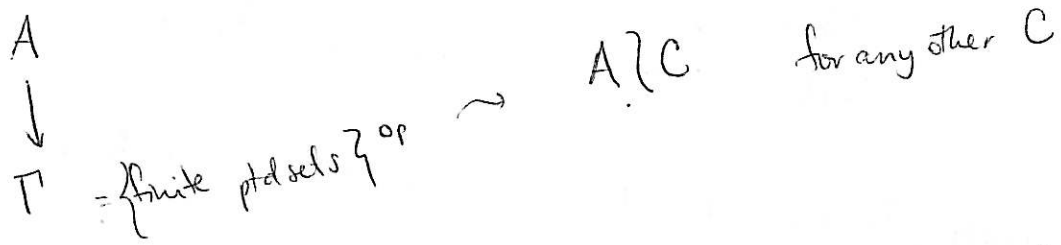
$\cdot \forall A \in \mathcal{Q} \quad \text{localization Seg}(A) \xrightarrow{C/A} T(A)$

$(\hat{\mathcal{Q}}_T, W_J) \underset{\mathcal{Q}}{\sim} (\hat{\mathcal{Q}}_T \times \Delta_1, W_R)$

Iterative procedure for n-categories

Part II

Wreath products



Ex.  $\cdot \mathcal{G}_1 \rightarrow T$   
 $\mathcal{G}_1 \int A$  full subcategory of the category of graphs enriched in  $\hat{A}$ .  
 $\Delta_0 = \{X\} \quad \Delta_1(X) = \begin{matrix} 0 & \xrightarrow{X} & 1 \end{matrix}$  for  $X \in \text{representable presheaf}$ .

$\hat{\mathcal{G}}_1 \int A = \text{graph}(\hat{A})$   
 $= \hat{A}$ -enriched graphs

$\hat{\mathcal{C}} \supset T \quad \mathcal{G}_1 \int C \supset T_1 \int T$  s.t.  $T_1 \int T\text{-Alg} = \text{Categories enriched in } T\text{-algebras} = \text{Cat}(T\text{-Alg})$ .

$\cdot \Delta \int A$  full subset of  $\text{Cat}(\hat{A})$  spanned by objects of shape



free category generated by  $\Delta_n(a_1, \dots, a_n)$  (as presheaves)  
 $\text{Hom}(0, 2) = a_1 \times a_2$   
 $\text{Hom}(0, k) = a_1 \times \dots \times a_k$

$\hat{A} \supset \text{id}$

$T_1 \text{id}$

$$\textcircled{H} T_1 \text{id}_{\hat{A}} = \Delta \int A$$

$$N: \text{Cat}(\hat{A}) \longrightarrow \Delta \int \hat{A}$$

Thm (C. Berger)  $\Delta \int \textcircled{H}_n = \textcircled{H}_{n+1}$

A small category. We have a Joyal localizer  $W_J, T_1 \text{id}_{\hat{A}}$  on  $\hat{A}$ .  
Consider an accessible localizer on  $\hat{A}$ . Assume  $W$  is stable by finite products

The regular localizer generated by

$$W_{J, T_1 \text{id}_{\hat{A}}} \Delta_!(X) \rightarrow \Delta_!(Y) \quad \text{for } X \rightarrow Y \text{ in } W$$

and by

defines the homotopy theory of  $(\infty, 1)$ -categories enriched in (fibrant) presheaves  $\text{ohf}$ .

Apply this for  $A = \textcircled{H}_n, W = W_{J, T_n}$ , this defines a model category structure on  $\hat{\textcircled{H}}_{n+1}$

Thm This structure is the same as the one associated to  $W_{J, T_n}$

$$\begin{array}{ccc} \hat{A} & \longrightarrow & \Delta \int \hat{A} \\ X & \longmapsto & 0 \xrightarrow{x} 1 \end{array}$$

is a left Quillen functor and ~~it is~~ right adjoint to the functor  $(X, a, b) \longmapsto X(a, b)$

Groupoid-like algebras

$$\begin{array}{ccc} \text{Localization} & S \subset C & \\ \downarrow & \downarrow & \\ \text{Groupoids} & S^{-1}S \longrightarrow S^{-1}C & \\ \longleftarrow & \text{Cat} & \end{array}$$

enough to know how to localize  $S^{-1}S$ .  
has left adj., right adj. (localize / maximal out)

Assume that we have the following assumptions:

$$1) A \in \mathcal{H}_0, \quad |NC/A| \simeq *$$

2  $\mathcal{H}_T$  is a local test category.

$A$  is a local test category if  $\exists I$  interval st  $\forall a \in A$

$$|A / (a \times I)| \simeq pt$$

$\Leftrightarrow \hat{A}$  has a model structure  $\sim Top / |A|$

All the examples given satisfy these conditions:  $\mathcal{H}_n, \Omega$

$W_k$  is a regular localizer of  $\hat{\mathcal{H}}_T$  generated by maps between representable presheaves

Prop  $W_{J,T} \subset W_k$

$$WT-Alg = DK(\hat{\mathcal{H}}_T, W_J)$$

Weak T-algebras (Dwyer-Kan localization)

$$grWT-Alg = DK(\hat{\mathcal{H}}_T, W_k) \sim spaces / B\mathcal{H}_T$$

groupoid-like

$$grWT-Alg \overset{\text{left-adjoint}}{\longleftarrow} \underset{i}{\longrightarrow} WT-Alg$$

Prop the functor  $i$  also has a right adjoint ("maximal subgroupoid-like")

