DIFFERENTIAL HOMOLOGICAL ALGEBRA 
AND HOMOGENEOUS SPACES

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Contents

Introduction
0. Recollections on differential modules

I. Differential graded homological algebra and Cotor
1. Injective classes; skeletally filtered comodules
2. Injective resolutions and derived functors
3. Categories of differential and filtered objects over a differential category
4. Generalities on differential derived functors
5. Remarks on the differential Cotor functor
6. Spectral properties of the cotensor product of skeletally filtered comodules

II. Algebras, coalgebras and adjointness
1. Twisting morphisms
2. Derivations on algebras and coalgebras
3. The classifying construction and loop construction
4. Homotopy properties of the adjunction morphisms
5. Acyclicity of the universal bundles and extended loop classifying adjointness

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III. Simplicial sets and fibrations
1. Simplicial sets, Map(X, Y) and homotopies
2. Coalgebra and comodule structures on chain modules
3. The category Map(Δ(set))
4. Calculation of the $E^1$-term of $C_*(\pi; M)$
5. The spectral sequence of a fibration
6. Remarks on various types of fibrations

IV. The collapsing theorem
1. Cones and cocones
2. Splitting theorem for algebras
3. Splitting theorem for coalgebras
4. Further properties of cofibrations of coalgebras
5. Quasi-Hopf algebras and quasi-commutative coalgebras
6. Study of the coalgebra $BC_*(T)$ for a torus $T$
7. Homological properties of a subtorus of a compact Lie group
8. The collapsing theorem

Introduction

Many of the interesting spaces in topology, for example spheres and Grassmann manifolds, are homogeneous spaces $G/U$, where $G$ and $U$ are compact Lie groups. Although the space $G/U$ is defined as a quotient of $G$, it has the homotopy type of the fibre of the fibration $BU \to BG$ of the classifying spaces induced by the inclusion $U \to G$. A basic problem is to calculate the homology $H_*(G/U)$ over a field in terms of morphism $H_*(BU) \to (BG)$ of coalgebras.

In [15], Eilenberg and Moore describe a general method for calculating the homology $H_*(F)$ of the fibre $F$ of a fibre map $E \to B$ in terms of the coalgebra morphism $H_*(E) \to H_*(B)$. Their method is to show that $H_*(F)$ is isomorphic to a differential Cotor functor and then in turn to approximate the differential derived functor by a spectral sequence of ordinary derived Cotor functors. The purpose of this article is to sharpen the technique of differential graded homological algebra in order to prove that this spectral sequence collapses for $G/U$ the fibre of $BU \to BG$ under suitable hypotheses. Roughly this is done by showing the existence of morphisms of differential coalgebras

$$H_*(BU) \to C_*(BU) \quad \text{and} \quad C_*(BG) \to H_*(BG)$$

which induce the identity on the homology level. This is not strictly possible, but it works in a suitable sense so that one can deduce the collapsing of this spectral sequence of Eilenberg and Moore.

In Part I the foundations of differential graded homological algebra are reconsidered and augmented from that which is given in [3] and [19]. These considerations are applied in Part II, where the relation between the categories of differential algebras and differential coalgebras are studied in order to analyze $C_*(BG)$. The relation between the geometry and the differential Cotor is extended in Part III beyond that.
considered in [5]. Finally, the collapsing of the spectral sequence in the case of a homogeneous space $G/U$ is derived under suitable hypotheses.

The first result in this direction was obtained for real coefficients in the context of cohomology by H. Cartan (see [8, 8a]), and in the case where $U$ and $G$ have the same rank by A. Borel (see [4]). Both of their techniques involved finding cochain representatives for cohomology classes which preserved the cup product algebra structure.

In the context of the collapsing of the spectral sequence introduced by Eilenberg and Moore, the problem was taken up by Paul Baum in his 1963 Princeton thesis [2] and later by Peter May [22]. Baum solved the problem in some special cases, and May announced the result for the case in which the groups have no $p$-torsion, where $p$ is the characteristic of the coefficient field for the homology groups. Recently, J. Wolf in his 1973 Brown thesis [30] and Hans J. Munkholm [27] have presented proofs of the collapsing of the spectral sequence.

0. Recollections on differential modules

0.1. Let $R$ be a commutative ring. Recall that a differential $R$-module $M$ is a graded $R$-module, also denoted by $M$, and a differential $d(M)_n : M_n \to M_{n-1}$, which is a morphism of $R$-modules, such that

$$d(M)_{n-1} d(M)_n = 0$$

for each integer $n$. A morphism $f : L \to M$ of differential $R$-modules is a morphism of graded modules such that

$$d(N) f = f d(M),$$

that is, $f = (f_n)$, where $f_n : L_n \to M_n$ is $R$-linear and

$$d(N)_n f_n = f_{n-1} d(M)_n$$

for each integer $n$. The composite of two morphisms of differential $R$-modules is just the composite of the morphisms of graded $R$-modules. A differential $R$-module $M$ is positive if $M_n = 0$ for $n < 0$ and negative if $M_n = 0$ for $n > 0$.

The category of differential $R$-modules and morphisms is denoted by $\mathcal{C}(R)$ and is an abelian category. We denote by $\mathcal{C}_+(R)$ the full subcategory of $\mathcal{C}(R)$ consisting of positive differential $R$-modules, and by $\mathcal{C}_-(R)$ the full subcategory consisting of negative differential $R$-modules. Both $\mathcal{C}_+(R)$ and $\mathcal{C}_-(R)$ are abelian categories. The natural functor from $\mathcal{C}(R)$ to $\text{Mod}(R)$, the category of graded modules, is exact and faithful. However, it is not full since morphisms of differential $R$-modules are those morphisms of the underlying graded $R$-modules which preserve differentials. Frequently we view $\text{Mod}(R)$ as a subcategory of $\mathcal{C}(R)$ consisting of $M$ with $d(M) = 0$. It is a full subcategory.
The suspension functor $s : \mathcal{C}(R) \to \mathcal{C}(R)$ is an automorphism of the category \(\mathcal{C}(R)\) defined on objects by the relations $s(M)_n = M_{n-1}$ and $d(s(M))_n = -d(M)_{n-1}$. Its inverse $s^{-1}$ is called the desuspension functor, and by iteration $s^q : \mathcal{C}(R) \to \mathcal{C}(R)$ is defined for each integer $q$. For $q$ positive, $s^q$ preserves $\mathcal{C}_+(R)$ but not $\mathcal{C}_-(R)$, and for $q$ negative, $s^q$ preserves $\mathcal{C}_-(R)$ but not $\mathcal{C}_+(R)$.

The tensor product on graded modules extends to the tensor product $$(\otimes) : \mathcal{C}(R) \times \mathcal{C}(R) \to \mathcal{C}(R)$$
of differential $R$-modules, where $$d(L \otimes M) = d(L) \otimes 1 + 1 \otimes d(M)$$
with the usual sign conventions concerning the tensor products of morphisms of degree $-1$. The tensor product on $\mathcal{C}(R)$ induces a tensor product functor on $\mathcal{C}_+(R)$ and on $\mathcal{C}_-(R)$, i.e. $\mathcal{C}_+(R)$ and $\mathcal{C}_-(R)$ are closed under tensor products.

0.2. Coalgebras. A coalgebra (positive differential graded coalgebra) $C$ is a positive differential $R$-module $C$ together with morphisms $\Delta(C) : C \to C \otimes C$ and $\epsilon(C) : C \to R$ such that $\Delta(C)$ is associative and $\epsilon(C)$ is a unit for the comultiplication (or diagonal) $\Delta(C)$ of $C$. For two coalgebras $C', C''$, a morphism $f : C' \to C''$ is a morphism in $\mathcal{C}_+(R)$ such that $$(f \otimes f) \Delta(C') = \Delta(C'') f, \quad \epsilon(C'') f = \epsilon(C').$$
Composition in $\mathcal{C}_+(R)$ induces composition of morphisms of coalgebras. Observe that $R$ admits a unique structure of a coalgebra.

A supplemented coalgebra $C$ is a coalgebra $C$ together with a morphism of coalgebras $\eta(C) : R \to C$. Note that $\epsilon(C) \eta(C) = R$, the identity on $R$, since $R$ is the terminal point of the category of coalgebras. A morphism $f : C' \to C''$ of supplemented coalgebras is a morphism of coalgebras such that $\eta(C'') f = \eta(C')$. For a natural number $n$, the coalgebra $C$ is $n$-connected if $\epsilon(C)_q$ is an isomorphism for $q \leq n$. The 0-connected coalgebras are called just connected coalgebras, and they admit a unique supplemented structure. The 1-connected coalgebras are also called simply connected.

If $C$ is a coalgebra, then the graded module $Z'(C)$, where $$Z'(C)_n = \text{coker}(d(C)_{n+1} : C_{n+1} \to C_n),$$
admits a unique coalgebra structure such that the natural morphism $C \to Z'(C)$ in $\mathcal{C}_+(R)$ is a morphism of coalgebras. Indeed, suppose one considers the full subcategory of the category of coalgebras determined by those coalgebras $C$ with $d(C) = 0$, then $Z'$ is a functor from the category of coalgebras to this full category which is a coadjoint for the inclusion functor from the subcategory to the whole category. Observe also that if $C$ is a supplemented coalgebra, then $Z'(C)$ admits a unique supplemented coalgebra structure such that $C \to Z'(C)$ is a morphism of supplemented coalgebras. If $C$ is $n$-connected, then $Z'(C)$ is also $n$-connected.
0.3. Comodules. Let $C$ be a coalgebra. A left $C$-comodule $Y$ is a positive differential $R$-module $Y$ together with a morphism $\Delta(Y) : Y \to C \otimes Y$ such that

$$(C \otimes \Delta(Y)) \Delta(Y) = (Y \otimes \Delta(C)) \Delta(Y), \quad (\epsilon(C) \otimes Y) \Delta(Y) = Y;$$

i.e. such that the cooperation $\Delta(Y)$ of $C$ on $Y$ is associative and unital. If $Y', Y''$ are two left $C$-comodules, a morphism $g : Y' \to Y''$ is a morphism in $\mathcal{C}_+(R)$ such that

$$(C \otimes g) \Delta(Y') = \Delta(Y'') g.$$ 

The category of left $C$-comodules is denoted $\mathcal{C} \text{Co} \mathcal{M}$. It is an additive category with cokernels such that the natural forgetful functor to $\mathcal{C}_+(R)$ preserves addition and cokernels. It is abelian if $C$ is $R$-flat, and in this case the natural functor to $\mathcal{C}_+(R)$ is exact. The category $\mathcal{C} \mathcal{M} \mathcal{C}$ of right $C$-comodules is defined similarly and has similar properties.

0.4. Algebras. An algebra (positive differential graded algebra) $A$ is a positive differential $R$-module $A$ together with morphisms $\phi(A) : A \otimes A \to A$ and $\eta(A) : R \to A$ such that $\phi(A)$ is associative and $\eta(A)$ is a unit for the multiplication $\phi(A)$ of $A$. For two algebras $A', A''$, a morphism $f : A' \to A''$ is a morphism in $\mathcal{C}_+(R)$ such that

$$\phi(A'')(f \otimes f) = f\phi(A'), \quad f\eta(A') = \eta(A'').$$

Composition in $\mathcal{C}_+(R)$ induces composition of morphisms of algebras. Observe that $R$ admits a unique structure of an algebra.

A supplemented algebra $A$ is an algebra $A$ together with a morphism of algebras $\epsilon(A) : A \to R$. Note that $\epsilon(A) \eta(A) = R$ since $R$ is an initial point of the category of algebras. A morphism $f : A' \to A''$ of supplemented algebras is a morphism of algebras such that $\epsilon(A') = \epsilon(A'')$. For a natural number $n$, the algebra $A$ is $n$-connected if $\eta(A)_{q}$ is an isomorphism for $q \leq n$. The 0-connected algebras are called just connected algebras, and they admit a unique supplemented structure. The 1-connected algebras are also called simply connected.

If $A$ is an algebra, then the graded module $Z(A)$, where

$$Z(A)_n = \ker(d(A)_{n} : A_n \to A_{n-1}),$$

admits a unique algebra structure such that the natural morphism $Z(A) \to A$ in $\mathcal{C}_+(R)$ is a morphism of algebras. Indeed, suppose one considers the full subcategory of the category of algebras determined by those algebras $A$ with $d(A) = 0$; then $Z$ is a functor from the category of algebras to this full subcategory which is an adjoint for the inclusion functor from the subcategory to the whole category. Observe also that if $A$ is a supplemented algebra, then $Z(A)$ admits a unique supplemented algebra structure such that $Z(A) \to A$ is a morphism of supplemented algebras. If $A$ is $n$-connected, then $Z(A)$ is also $n$-connected.

0.5. Modules. Let $A$ be an algebra. A right $A$-module $L$ is a positive differential $R$-module $L$ together with a morphism $\phi(L) : L \otimes A \to L$ such that

$$\phi(L)(\phi(L) \otimes A) = \phi(L)(L \otimes \phi(A)), \quad \phi(L)(L \otimes \eta(A)) = L.$$
i.e. such that the operation $\phi(L)$ of $A$ on $L$ is associative and unitary. If $L', L''$ are right $A$-modules, a morphism $f : L' \to L''$ is a morphism in $\mathcal{C}_+(R)$ such that

$$\phi(L'')(f \circ A) = f\phi(L').$$

The category of right $A$-modules is denoted $\mathcal{M}_A$. It is an abelian category such that the natural forgetful functor to $\mathcal{C}_+(R)$ is exact. The category $\mathcal{M}_A$ of left $A$-modules is defined similarly and has similar properties.

0.6. Filtered differential modules. Recall that a filtered differential $R$-module $M$ consists of a differential $R$-module $F_\infty M$ and a family of subobjects $(F_pM)$ of $F_\infty M$ such that

$$F_pM \subset F_{p+1}M$$

for all integers $p$. Frequently we will denote $F_\infty M$ simply by $M$. A filtered differential $R$-module $M$ is complete if

$$F_\infty M = \bigcup_p F_pM.$$

Henceforth in this paper, filtered object will mean complete filtered object. A filtered object is positively filtered if $F_pM = 0$ for $p < 0$.

For two filtered differential $R$-modules $M'$ and $M''$ a morphism $f : M' \to M''$ is a morphism $F_\infty f : F_\infty M' \to F_\infty M''$ of differential $R$-modules such that

$$(F_\infty f)(F_pM') \subset F_pM'.$$

Usually $(F_\infty f) | F_pM'$ is denoted by $F_pf : F_pM' \to F_pM''$. The category of filtered differential $R$-modules is an additive category denoted by $\mathcal{F}_C(R)$. The full subcategory determined by the positively filtered differential $R$-modules is denoted by $\mathcal{F}_+(R)$. The categories $\mathcal{F}_+(R)$, $\mathcal{F}_-(R)$, $\mathcal{F}_+(R)$ and $\mathcal{F}_-(R)$ are defined in the obvious way as full subcategories of $\mathcal{F}_R(R)$.

The category $\mathcal{F}_R(R)$ is a category with a tensor product. Indeed, if $M, N$ are objects of $\mathcal{F}_R(R)$, then $M \otimes N$ is the object such that

$$F_\infty(M \otimes N) = F_\infty M \otimes F_\infty N,$$

and

$$F_p(M \otimes N) = \bigcap_{r+s=p} F_rM \otimes F_sN \to F_rM \otimes F_sN.$$

The tensor product on $\mathcal{F}_R(R)$ is associative, commutative, and it has a unit $R$, where

$$F_pR = \begin{cases} F_\infty R = R & \text{for } p \geq 0, \\ 0 & \text{for } p < 0. \end{cases}$$

The full subcategories $\mathcal{F}_+(R)$ and $\mathcal{F}_-(R)$ are closed under tensor products.

An object $M$ of $\mathcal{F}_R(R)$ is regularly filtered if $(F_pM)_n$ (or $F_pM_n$) = $(F_\infty M)_n$ for $n < p$. If $M$ and $N$ are regularly filtered, then so also is $M \otimes N$. An object $M$ of $\mathcal{F}_R(R)$ is filtered by skeletons if
\[ F_p M_n = \begin{cases} F_\infty M_n & \text{for } n \leq p, \\ 0 & \text{for } n > p. \end{cases} \]

One may identify \( \mathcal{C}(R) \) with the full subcategory of \( \mathcal{FC}(R) \) determined by the objects filtered by skeletons. This identification is compatible with tensor products. Let \( \mathcal{FC}(R) \) denote the full subcategory of \( \mathcal{FC}(R) \) determined by regularly filtered objects.

The category \( \mathcal{FC}(R) \) is not abelian, but it does have kernels and cokernels. Indeed suppose \( f : M' \rightarrow M'' \) is a morphism in \( \mathcal{FC}(R) \). If \( f' : N \rightarrow M' \) is the kernel of \( f \), then \( F_\infty f' \) is the kernel of \( F_\infty f \), and

\[ F_p N = F_\infty N \cap F_p M'. \]

If \( f'' : M'' \rightarrow C \) is the cokernel of \( f \), then \( F_\infty f'' \) is the cokernel of \( F_\infty f \), and

\[ F_p M'' = \text{Im}(F_p M'' \rightarrow F_\infty C). \]

Thus \( F_\infty : \mathcal{FC}(R) \rightarrow \mathcal{C}(R) \) preserves kernels and cokernels. For \( p \geq 0 \), the functor \( F_p : \mathcal{FC}(R) \rightarrow \mathcal{C}(R) \) is additive, but it does not in general preserve either kernels or cokernels.

A sequence
\[ M' \xrightarrow{f'} M \xrightarrow{f''} M'' \]
in \( \mathcal{FC}(R) \) is \textit{filtering exact} (or strict exact, see [26]) if
\[ F_p M' \rightarrow F_p M \rightarrow F_p M'' \]
is exact for all integers \( p \); longer sequences are filtering exact if all of their two-term components are filtering exact. Note in particular that if
\[ 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \]
is a short filtering exact sequence, then
\[ 0 \rightarrow E^0(M') \rightarrow E^0(M) \rightarrow E^0(M'') \rightarrow 0 \]
is a short exact sequence, because for an integer \( p \) there is a commutative diagram (note that \( E^0_p(M) = F_p M/F_{p-1} M \))
The iterated suspension functor \( s^n : \mathcal{C}(R) \to \mathcal{C}(R) \) extends to a suspension automorphism \( s^n : \mathcal{F}\mathcal{C}(R) \to \mathcal{F}\mathcal{C}(R) \), where for a filtered differential \( R \)-module \( \mathcal{F} \) the relations
\[
F_\infty s^n M = s^n F_\infty M, \quad F_p s^n (M) = s^n F_{p-n} (M)
\]
hold. The inclusion functor defined by using the skeleton filtration \( \mathcal{C}(R) \to \mathcal{F}\mathcal{C}(R) \) and \( F_\infty : \mathcal{F}\mathcal{C}(R) \to \mathcal{C}(R) \) commute with the action of \( s^n \) for each integer \( n \).

The above definitions of differential graded object apply to any additive category \( \mathcal{A} \), and the following categories \( \mathcal{E}(\mathcal{A}) \), \( \mathcal{E}_+(\mathcal{A}) \), and \( \mathcal{E}_-(\mathcal{A}) \) can be formed. (In fact, \( \mathcal{A} \) need only be a pointed category). For an abelian category \( \mathcal{A} \), \( \mathcal{F}\mathcal{C}(\mathcal{A}) \) and the various full subcategories are defined as for the case of \( \mathcal{A} = \mathcal{C}(R) \), the category of graded \( R \)-modules. We can consider \( \mathcal{A} \) to be the intersection of \( \mathcal{E}_+(\mathcal{A}) \) and \( \mathcal{E}_-(\mathcal{A}) \).

0.7. Hom functor and homotopies. For an additive category \( \mathcal{A} \), the homomorphism functor
\[
[ , ] : \mathcal{C}(\mathcal{A})^\bullet \times \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathbb{Z})
\]
is defined by the relations
\[
[X, Y]_n = \prod_i \mathcal{A}(X_i, Y_{i+n}),
\]
the product over all integers, and
\[
D : [X, Y]_n \to [X, Y]_{n-1}
\]
is defined by
\[
Df = d(Y)f - (-1)^n fd(X).
\]
Observe that the morphism group \( \mathcal{C}(\mathcal{A})(X, Y) \) is \( \mathbb{Z}[X, Y]_0 \) since \( Df = 0 \) means \( d(Y)f = fd(X) = 0 \). Two morphisms \( f, g : X \to Y \) in \( \mathcal{C}(\mathcal{A}) \) are homotopic provided there is a homotopy \( s \in [X, Y]_{+1} \) with
\[
f - g = Ds = d(Y)s + sd(X).
\]

Two morphisms being homotopic is an additive, natural equivalence relation on \( \mathcal{C}(\mathcal{A})(X, Y) \). The quotient category \( \mathcal{E}(\mathcal{A}) \) under the relation of homotopy on the morphisms is an additive category, and \( \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A}) \) is an additive functor. We also have the quotients \( \mathcal{E}_+(\mathcal{A}) \to \mathcal{E}_+(\mathcal{A}) \) and \( \mathcal{E}_-(\mathcal{A}) \to \mathcal{E}_-(\mathcal{A}) \) of two full subcategories. Morphisms of degree \( n \) have a meaning in \( \mathcal{C}(\mathcal{A}) \); namely, they are elements of the group \( H[X, Y]_n \). Finally, the morphism group \( \mathcal{C}(\mathcal{A}) (X, Y) = H_0 [X, Y] \).
1. Differential graded homological algebra and Cotor

In this part we review and extend the generalities connected with differential graded homological algebra which apply to Cotor, the derived functor of the cotensor product. These notions have been discussed in [13] and [19]. In particular we distinguish between differential derived functors of the first and second kind. The new material in this part will be used in Part II to study the adjoint pair relating algebras and coalgebras and in Part III to relate the functor Cotor to the homology of induced fibrations.

1. Injective classes; skeletally filtered comodules

A pair of morphisms $X' \to X \to X''$ in a pointed category is called a (two-term) sequence provided the composite is zero. Now recall the following definition.

1.1. Definition. An injective class $\mathcal{G}$ in a pointed category $\mathcal{X}$ is a pair $(\mathcal{G}, \text{Seq}(\mathcal{G}))$, where $\mathcal{G}$ is a class of objects and $\text{Seq}(\mathcal{G})$ is a class of two-term sequences in $\mathcal{X}$ such that:

(1) An object $I$ in $\mathcal{X}$ is in $\mathcal{G}$ if and only if for each $X'' \to X \to X'$ in $\text{Seq}(\mathcal{G})$ the sequence

$$\mathcal{X}(X', I) \to \mathcal{X}(X, I) \to \mathcal{X}(X'', I)$$

of pointed sets is exact.

(2) A sequence $X'' \to X \to X'$ in $\mathcal{X}$ is in $\text{Seq}(\mathcal{G})$ if and only if for each $I \in \mathcal{G}$ the sequence

$$\mathcal{X}(X', I) \to \mathcal{X}(X, I) \to \mathcal{X}(X'', I)$$

of pointed sets is exact.

(3) For each morphism $X \to Y$ in $\mathcal{X}$ there exists $I \in \mathcal{G}$ and a morphism $Y \to I$ such that $X \to Y \to I$ is in $\text{Seq}(\mathcal{G})$.

1.2. Example. Let $\mathcal{X}$ be a pointed category with cokernels. The split injective class $\mathcal{G}_s$ has $\mathcal{G}_s$ equal to all objects of $\mathcal{X}$ and $\text{Seq}(\mathcal{G})$ equal to all $X' \to X \to X''$ such that 1.1(2) holds; in other words, those sequences such that for the factorization $X \to C \to X'$ of $u$ by the cokernel $c : X \to C$ of $u$, the morphism $w$ is a split monomorphism, i.e. there exists $r : X' \to C$ with $rw = C$.

An object $W$ is a retract of $X$ in $\mathcal{X}$ provided there exist $u : W \to X$ and $v : X \to W$ with $uw = W$. For a class $\mathcal{E}$ of objects in $\mathcal{X}$, let $\text{ret}(\mathcal{E})$ denote the class of all retracts of objects in $\mathcal{E}$. Note that an object isomorphic to an object in $\mathcal{E}$ is in $\text{ret}(\mathcal{E})$.

For the next result see [3, p. 15] or [19, p. 400].
1.3. Adjoint functor theorem. Let $\mathcal{X}$ and $\mathcal{Y}$ be pointed categories, and let $S \rightarrow T : (\alpha, \beta)$ be an adjoint pair, where $T : \mathcal{X} \rightarrow \mathcal{Y}$ and $S : \mathcal{Y} \rightarrow \mathcal{X}$. If $\mathcal{I}$ is an injective class in $\mathcal{X}$, then $(\text{ret}(T(\mathcal{I})), S^{-1}(\text{Seq}(\mathcal{I})))$ is an injective class in $\mathcal{Y}$.

1.4. Example 1. Let $V : c\text{CoMod} \rightarrow \mathcal{C}_+^+(R)$ be the forgetful functor which assigns to a comodule $M$ its underlying differential $R$-module and
\[
C \otimes ( ) : \mathcal{C}_+^+(R) \rightarrow c\text{CoMod}
\]
the extended comodule functor, where
\[
V(C \otimes X) = C \otimes X
\]
in $\mathcal{C}_+^+(R)$ and
\[
\Delta(C \otimes X) = \Delta(C) \otimes X.
\]
The usual adjunction morphism
\[
\alpha : V(C \otimes ( )) \rightarrow 1
\]
is the tensor product of $\tau(C) : C \rightarrow P$ with $X$, and
\[
\beta : 1 \rightarrow (C \otimes ( ))V
\]
is the structure morphism $\Delta(M) : M \rightarrow C \otimes M$. We apply the adjoint functor theorem to $V \leftarrow C \otimes ( )$ and the split injective class $\mathcal{I}_e$ in $\mathcal{C}_+^+(R)$. With 1.3 this defines the extended injective class $\mathcal{I}_e$ in $c\text{CoMod}$, where the injectives in $\mathcal{I}_e$ are retracts of extended $C$-comodules $C \otimes X$, and the short exact sequences in $\text{Seq}(\mathcal{I}_e)$ are those $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ which split in $\mathcal{C}_+^+(R)$.

This example extends to certain filtered comodules which will arise naturally in a geometric context, see Part III. Let $C$ be a coalgebra, and give it the skeleton filtration in the following considerations. Recall that the skeleton filtration is defined by
\[
(F_i C)_i = \begin{cases} 
C_i & \text{for } i \leq p, \\
0 & \text{for } i > p.
\end{cases}
\]

1.5. Definition. A skeletally filtered left $C$-comodule $M$ is a positive regularly filtered positive differential $R$-module $M$ together with a morphism $\Delta(M) : M \rightarrow C \otimes M$ of filtered differential $R$-modules such that neglecting filtrations $(M, \Delta(M))$ is a left $C$-comodule. For two skeletally filtered left $C$-comodules $M'$ and $M''$, a morphism $f : M' \rightarrow M''$ is a morphism in $\mathcal{F}_c(\mathcal{C}_+^+(R))$ which preserves the left $C$-comodule structure.

Let $\mathcal{F}_c\text{CoMod}$ denote the category of skeletally filtered left $C$-comodules. It is an additive category, and
\[
F_\omega : \mathcal{F}_c\text{CoMod} \rightarrow c\text{CoMod}
\]
is an additive functor preserving cokernels. Similarly, the category \( \mathcal{S} \mathcal{T} \mathcal{C} \mathcal{M}_C \) of skeletally filtered right \( C \)-comodules is defined. Let \( \mathcal{R} \mathcal{T} \mathcal{C}(R) \) denote the category \( \mathcal{R} \mathcal{T} \mathcal{C}(R) \cap \mathcal{F}_+ \mathcal{C}_+(R) \) of positive regularly filtered positive differential \( R \)-modules.

1.6. Example II. Let

\[ V : \mathcal{S} \mathcal{T} \mathcal{C} \mathcal{C} \mathcal{O} \mathcal{M} \to \mathcal{R} \mathcal{T} \mathcal{C}_+(R) \]

be the forgetful functor which assigns to a skeletally filtered (left) \( C \)-comodule \( M \) its underlying regularly filtered differential \( R \)-module, and

\[ C \otimes ( ) : \mathcal{R} \mathcal{T} \mathcal{C}_+(R) \to \mathcal{S} \mathcal{T} \mathcal{C} \mathcal{M} \]

the extended comodule functor, where

\[ V(C \otimes X) = C \otimes X \]

in \( \mathcal{R} \mathcal{T} \mathcal{C}_+(R) \) and

\[ \Delta(C \otimes X) = \Delta(C) \otimes X. \]

As in 1.4, the adjunction morphism

\[ \alpha : V(C \otimes ( )) \to 1 \]

is the tensor product of \( \eta(C) : C \to R \) with \( X \) and

\[ \beta : 1 \to (C \otimes ( ))^V \]

is the structure morphism \( \Delta(M) : M \to C \otimes M \). Applying the adjoint functor theorem 1.3 to \( V \dashv C \otimes ( ) \) and the split injective class \( \mathcal{I}_C \) in \( \mathcal{C}_+(R) \), we define the extended injective class \( \mathcal{I}_C \) in \( \mathcal{S} \mathcal{T} \mathcal{C} \mathcal{M} \), where the injectives in \( \mathcal{I}_C \) are retracts of extended \( C \)-comodules \( C \otimes X \), and the short exact sequences in \( \text{Seq}(\mathcal{I}_C) \) are those

\[ 0 \to M' \to M \to M'' \to 0 \]

which split in \( \mathcal{R} \mathcal{T} \mathcal{C}_+(R) \).

1.7. Remark. Clearly the two examples 1.4 and 1.6 are related. For example, the following diagrams are commutative:

\[
\begin{array}{ccc}
\mathcal{S} \mathcal{T} \mathcal{C} \mathcal{M} & \xrightarrow{F_0} & \mathcal{C} \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{F}_+ \mathcal{C}_+(R) & \xrightarrow{F_0} & \mathcal{C}_+(R)
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{S} \mathcal{T} \mathcal{C} \mathcal{M} & \xrightarrow{F_0} & \mathcal{C} \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{F}_+ \mathcal{C}_+(R) & \xrightarrow{F_0} & \mathcal{C}_+(R)
\end{array}
\]

For \( X \) in \( \mathcal{R} \mathcal{T} \mathcal{C}_+(R) \) it follows that \( \alpha(F_0 X) = F_0 \alpha(X) \), and for \( M \) in \( \mathcal{S} \mathcal{T} \mathcal{C} \mathcal{M} \) it follows that \( F_0 \beta(M) = \beta F_0(M) \).

The discussion in 1.6 and 1.7 also applies to right \( C \)-comodules.
1.8. Spectral properties. For a coalgebra $C$ filtered by skeletons

$$E^0(C)_{p,q} = \begin{cases} C_p & \text{for } q = 0, \\ 0 & \text{for } q \neq 0, \end{cases}$$

with $d^0(C) = 0$, and

$$E^1(C)_{p,q} = \begin{cases} C_p & \text{for } q = 0, \\ 0 & \text{for } q \neq 0, \end{cases}$$

with $d^1(C)_{p,0} = d(C)_{p,0}$. Hence we can identify $E^1(C)$ with $C$ since $E^1(C)_{*,0} = C_0$. Assume now that $C$ is an $R$-flat coalgebra. For $X$ in the category $\mathcal{R}\mathcal{H}_*(R)$ we have $E^1(C \otimes X) = C \otimes E^1(X)$, and for any skeletally filtered $C$-comodule $M$ the comultiplication $\Delta(M) : M \to C \otimes M$ induces a left $C$-comodule structure on $E^1(M)$.

2. Injective resolutions and derived functors

Using the homotopy category $\mathcal{C}_-(\mathcal{A})_\pi$ of negative complexes over an additive category, we can interpret the basic comparison theorem for injective resolutions; see [19, Theorem (2.2)]. The dual picture for projective resolutions and $\mathcal{C}_+(\mathcal{A})_\pi$ is left to the reader.

2.1. Existence and comparison of injective resolutions. Let $\mathcal{D}$ be an injective class in an additive category $\mathcal{A}$, and let $i : \mathcal{A} \to \mathcal{C}_-(\mathcal{A})_\pi$ denote the natural inclusion functor. Then there is a pair $(r, \gamma)$, where $r : \mathcal{A} \to \mathcal{C}_-(\mathcal{A})_\pi$ is a functor and $\gamma : i \to r$ a morphism of functors, such that

$$0 \to A \xrightarrow{\gamma(A)} r(A)_0 \to r(A)_{-1} \to \cdots \to r(A)_m \to \cdots$$

is an $\mathcal{D}$-injective resolution, i.e. all $r(A)$ are in $\mathcal{D}$ and the sequence is $\mathcal{D}$-exact. Moreover, if $(r', \gamma')$ is a second such $\mathcal{D}$-injective resolution pair, then there is a unique morphism $\lambda : r \to r'$ of functors such that $\lambda \gamma = \gamma'$ and in addition $\lambda$ is an isomorphism.

A resolution functor is a pair $(r, \gamma)$, where $r : \mathcal{A} \to \mathcal{C}_-(\mathcal{A})_\pi$ is a functor and $\gamma : i \to r$ is a morphism of functors, such that $\gamma(A) : A \to r(A)_*$ is an $\mathcal{D}$-injective resolution of $A$ in $\mathcal{A}$ as in 2.1. Here the difference is that $r(A)$ is natural in $\mathcal{C}_-(\mathcal{A})_\pi$ instead of just the quotient homotopy category $\mathcal{C}_-(\mathcal{A})_\pi$. In an additive category with cokernels, a resolution functor can be defined by an $\mathcal{D}$-resolvent $(U, \beta)$, where $U : \mathcal{A} \to \mathcal{A}$ is a functor such that $U(A)$ is in $\mathcal{D}$ for all $A$ in $\mathcal{A}$ and $\beta : \mathcal{A} \to U$ is a morphism such that $0 \to A \xrightarrow{\beta(A)} U(A)$ is in Seq$(\mathcal{D})$ for all $A$. Then $\gamma(A) : A \to r(A)_*$ is defined by the standard splicing procedure, where $\kappa(A)$ is the cokernel of $\beta(A)$:

$$0 \to A \xrightarrow{\beta(A)} U(A) = r(A)_0 \xrightarrow{d_0} U(Z'(A)) = r(A)_{-1} \xrightarrow{d_{-1}} U(Z'(A)_{-1}) \xrightarrow{\beta} Z'(A) \xrightarrow{\beta} Z'(Z'(A))_0 = Z'(A)_{-1}$$
2.2. Definition. Let $S \rightarrow T : (\alpha, \beta)$ be an adjoint pair $T : \mathcal{X} \rightarrow \mathcal{A}$ and $S : \mathcal{A} \rightarrow \mathcal{X}$, where $\mathcal{A}$ and $\mathcal{X}$ are additive categories with cokernels. Let $\mathcal{G}_T = \text{ret}(T(\mathcal{G}_s))$ be the injective class in $\mathcal{A}$ coming from the split injective class $\mathcal{G}_s$ on $\mathcal{X}$. Then the canonical $\mathcal{G}_T$-resolution functor $(U_*, \beta)$, where $U_* : \mathcal{A} \rightarrow \mathcal{C}(\mathcal{A})$, is the $\mathcal{G}_T$-resolution functor defined by the resolvent $(\beta_S, \beta)$.

In fact, the adjoint pair $S \rightarrow T : (\alpha, \beta)$ gives more, namely, a triple $(TS, \beta, \lambda)$, where $\lambda : (TS)^2 \rightarrow TS$, which is used to define the standard $\mathcal{G}_T$-resolution functor and various simplicial constructions; for more details see [14].

In Examples 1.4 and 1.6, the functor $T(X) = C \otimes X$, and the injective class $\mathcal{G}_T$ is just the extended injective class $\mathcal{G}_e$, and the canonical resolution $\beta : 1 \rightarrow U_*$ is just the iteration as above of the resolvent $\beta : 1 \rightarrow (C \otimes (\_))V$. The two resolution sequences for $C \text{Co} \mathcal{M}$ and $\mathcal{F} C \text{Co} \mathcal{M}$ both denoted $(U_*, \beta)$ coming from this resolvent are related by

$$U_*(F_{\infty}(M)) = (\mathcal{C}_- F_{\infty}) U_*(M), \quad U_*(F_{\infty}(f)) = (\mathcal{C}_- F_{\infty}) U_*(f),$$

where $f : M \rightarrow M'$ is a morphism of skeletally filtered $C$-comodules. For an additive functor $T : \mathcal{A} \rightarrow \mathcal{B}$ we denote by

$$\mathcal{C}_-(T) : \mathcal{C}_-(\mathcal{A}) \rightarrow \mathcal{C}_-(\mathcal{B})$$

the canonical extension of $T$.

More generally, let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{A}$ be additive categories, and let $T : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{A}$ be an additive functor of two variables. The canonical extension

$$\mathcal{C}_-(T) : \mathcal{C}_-(\mathcal{X}) \times \mathcal{C}_-(\mathcal{Y}) \rightarrow \mathcal{C}_-(\mathcal{A})$$

of $T$ is defined by

$$\mathcal{C}_-(T)(X, Y)_n = \bigsqcup_{r+s=n} T(X_r, Y_s),$$

for the canonical injections

$$q_{r,s} : T(X_r, Y_s) \rightarrow \mathcal{C}_-(T)(X, Y)_{r+s}$$

the relation

$$d_{r+s} q_{r,s} = q_{r-1,s} T(d_r, Y_s) + (-1)^{\gamma} q_{r,s-1} T(X_r, d_s)$$

holds, and

$$\mathcal{C}_-(T)(f, g)_n = \bigsqcup_{r+s=n} T(f_r, g_s)$$

because these are finite coproducts. A similar extension

$$\mathcal{C}_+(T) : \mathcal{C}_+(\mathcal{X}) \times \mathcal{C}_+(\mathcal{Y}) \rightarrow \mathcal{C}_+(\mathcal{A})$$

can be defined.
2.3. Remarks. The canonical extension $E_\circ (T)$ preserves homotopies and thus induces an additive functor of two variables

$$E_\circ (T)_\pi : E_\circ (X)_\pi \times E_\circ (Y)_\pi \to E_\circ (A)_\pi.$$ 

Suppose $\mathcal{G} = (\mathcal{G}', \mathcal{G}'')$ is an injective class in $X \times Y$. There is a functor

$$r = r_1 \times r_2 : X \times Y \to E_\circ (X)_\pi \times E_\circ (Y)_\pi$$

induced by the injective class following 2.1. Let

$$T_r = E_\circ (T)_\pi r : X \times Y \to E_\circ (A)_\pi$$

be the composite functor.

2.4. Definition. For a functor $K : E_\circ (A)_\pi \to \mathcal{B}$ and an injective class $\mathcal{G} = (\mathcal{G}', \mathcal{G}'')$ in $X \times Y$, the derived functor of $T : X \times Y \to A$ relative to the injective class $\mathcal{G} = (\mathcal{G}', \mathcal{G}'')$ and the functor $K : E_\circ (A)_\pi \to \mathcal{B}$ is the composite

$$KT_r = K E_\circ (T)_\pi r : X \times Y \to \mathcal{B}.$$ 

For example, when $\mathcal{A}$ is abelian, $\mathcal{B}$ is the category of negatively graded $\mathcal{A}$-objects, and $K = H : E_\circ (\mathcal{A}) \to \mathcal{B}$ is the homology functor, then $HT_r = R_\circ T$, the right derived functor of $T$ relative to $\mathcal{G}$ in the usual sense.

Differential derived functors come up when $\mathcal{A}$ is itself a category of the form $E(\mathcal{B})$ for an abelian category $\mathcal{B}$. In the next section we study $E(E(\mathcal{B}))$ and various functors on this category. We find several advantages to using this category instead of the category of double complexes or bi-complexes.

3. Categories of differential and filtered objects over a differential category

Frequently in dealing with graded or differential objects we will take a countable product or coproduct, and it will be important that these operations be exact. Hence the following definition is useful.

3.1. Definition. An abelian category $\mathcal{A}$ is summable provided countable exact products exist, $\mathcal{A}$ is cosummable provided countable exact coproducts exist, and $\mathcal{A}$ is bisummable provided countable exact products and exact coproducts exist.

Observe that the category of left (or right) modules over a ring is bisummable.

For an abelian category $\mathcal{A}$ observe that an object $S$ of the category $E(E(\mathcal{A}))$ is three for each pair of integers $(p, q)$: an object $X_{p,q}$ of $\mathcal{A}$, and morphisms

$$d^1(X)_{p,q} : X_{p,q} \to X_{p-1,q}, \quad d^2(X)_{p,q} : X_{p,q} \to X_{p,q-1}$$
such that
\[ d_{p-1,q}^1 d_{p,q}^1 = 0, \quad d_{p,q-1}^2 d_{p,q}^2 = 0, \quad d_{p-1,q}^2 d_{p,q}^1 = d_{p,q-1}^1 d_{p,q}^2. \]
Similarly, a morphism \( f : X \to Y \) in \( C(C(\mathcal{A})) \) is a family \( f = (f_{p,q}) \) for each pair of integers \((p, q)\) such that
\[ d_{p,q}^1 f_{p,q} = f_{p-1,q} d_{p,q}^1(X)_{p,q}, \quad d_{p,q}^2 f_{p,q} = f_{p,q-1} d_{p,q}^2(X)_{p,q}. \]

3.2. Definition. The switching functor
\[ \sigma : C(C(\mathcal{A})) \to C(C(\mathcal{A})) \]
assigns to \( X \) the object \( \sigma(X) \), where
\[ \sigma(X)_{p,q} = X_{p,q}, \quad d_1 \sigma(X)_{p,q} = d_2(X)_{q,p}, \quad d_2 \sigma(X)_{p,q} = d_1(X)_{q,p}, \]
and to each morphism \( f : X \to Y \) the morphism \( \sigma(f) \), where
\[ \sigma(f)_{p,q} = f_{q,p}. \]

The functor \( \sigma \) is an involution, that is, \( \sigma \sigma \) is the identity on \( C(C(\mathcal{A})) \). We have several obvious subcategories of \( C(C(\mathcal{A})) \), namely,
\[ \mathcal{C}(\mathcal{C}(\mathcal{A})), \quad \mathcal{C}(\mathcal{C}(\mathcal{A}^*)), \quad \mathcal{C}(\mathcal{C}(\mathcal{A})), \quad \mathcal{C}(\mathcal{C}(\mathcal{A}^*)), \quad \mathcal{C}(\mathcal{C}(\mathcal{A}^*)), \quad \mathcal{C}(\mathcal{C}(\mathcal{A})). \]
Under \( \sigma \) this family of subcategories is carried onto itself; for example,
\[ \sigma : \mathcal{C}(\mathcal{C}(\mathcal{A})) \to \mathcal{C}(\mathcal{C}(\mathcal{A}^*)) \]
and
\[ \sigma : \mathcal{C}(\mathcal{C}(\mathcal{A}^*)) \to \mathcal{C}(\mathcal{C}(\mathcal{A}^*)) \]
are isomorphisms with inverse \( \sigma \). We imbed \( \mathcal{C}(\mathcal{A}) \) in \( \mathcal{C}(\mathcal{C}(\mathcal{A})) \) as
\[ \mathcal{C}(\mathcal{C}(\mathcal{A})) \cap \mathcal{C}(\mathcal{C}(\mathcal{A})), \]
that is, all objects \( X \) such that \( X_{p,q} = 0 \) for \( p \neq 0 \).

3.3. Definition. Let \( f, g : X \to Y \) be two morphisms in \( C(C(\mathcal{A})) \). A homotopy of the first kind from \( f \) to \( g \) is a homotopy \( D : X \to Y \) in \( C(\mathcal{X}) \), where \( \mathcal{X} \) is the category \( C(\mathcal{A}) \). The quotient of \( C(C(\mathcal{A})) \), under the relation of equivalence by homotopies of the first kind is denoted by \( C(C(\mathcal{A}^*))_\pi \).

A homotopy of the second kind from \( f \) to \( g \) is a homotopy of the first kind from \( \sigma(f) \) to \( \sigma(g) \) in \( C(C(\mathcal{A})) \). The quotient of \( C(C(\mathcal{A})) \) under the relation of equivalence by homotopies of the second kind is denoted \( C(C(\mathcal{A})_\pi) \).
Note that the switching isomorphism $\sigma : C(C(\mathcal{A})) \rightarrow C(C(\mathcal{A}))$ induces an isomorphism, also denoted $\sigma : C(C(\mathcal{A})) \rightarrow C(C(\mathcal{A}))$ by definition. Explicitly the homotopy of the first kind $D : X \rightsquigarrow Y$ from $f$ to $g$ is a family of morphisms $D_{p,q} : X_{p,q} \rightarrow Y_{p+1,q}$ such that

$$d^1(Y)_{p+1,q} D_{p,q} + D_{p-1,q} d^2(X)_{p,q} = f_{p,q} - g_{p,q},$$

$$d^2(Y)_{p+1,q} D_{p,q} = D_{p,q-1} d^2(X)_{p,q}.$$

The isomorphisms mentioned just before Definition 3.3 have various quotients; for example,

$$\sigma : C(C(\mathcal{A})) \rightarrow C(C(\mathcal{A})).$$

In Section 0 the category $\mathcal{C}(\mathcal{A})$ was introduced along with various full subcategories determined by objects with positivity conditions. A filtered object $X$ is cococomplete if $\lim_{\rightarrow} F_p(X) \rightarrow F_\infty(X)$ is an isomorphism, complete if $F_\infty(X) \rightarrow \lim_{\leftarrow} F_p(X)$ is an isomorphism, and bicomplete if it is both complete and cococomplete.

For the remainder of this section, $\mathcal{A}$ is a bissummarable additive category.

3.4. Definition. The assembly functor

$$A : C(C(\mathcal{A})) \rightarrow \mathcal{C}(\mathcal{A}).$$

is the functor which assigns to an object $X$ in $C(C(\mathcal{A}))$ the filtered differential object $A(X)$, where

$$F_p A(X)_n = \bigsqcup_{r \leq p} X_{r,n-r},$$

$$F_\infty A(X)_n = \bigsqcup_{r \leq p} X_{r,n-r} \oplus \bigsqcup_{p \leq r} X_{r,n-r},$$

$$p_{r,n-r} d(A(X)) = d^1(X)_{r+1,n-r} p_{r+1,n-r} + (-1)^r d^2(X)_{r,n-r+1} p_{r,n-r+1},$$

with $p_{i,j} : (F_\infty A(X))_{i+j} \rightarrow X_{i,j}$ the natural projection. If $f : X \rightarrow Y$ is a morphism in $C(C(\mathcal{A}))$, then

$$(F_p A(f))_n = \bigsqcup_{r \leq p} f_{r,n-r-1}, \quad F_\infty A(f)_n = \bigsqcup_{r \leq p} f_{r,n-r} \oplus \bigsqcup_{p < r} f_{r,n-r}.$$

Observe that $F_\infty A(X)_n$ is independent of $p$ in its definition and $i_p(A(X))_n : F_p A(X)_n \rightarrow F_\infty A(X)_n$ is just the natural injection into the coproduct, and the object $A(X)$ is bicomplete from elementary relations between limits and products and between colimits and coproducts. Further,

$$E^0_{p,q} A(X) = X_{p,q}, \quad d^0(A(X))_{p,q} = (-1)^p d^2(X)_{p,q}.$$
and \( d^1(A(X))_{p,q} \) is induced by \( d^1(X)_{p,q} \). Indeed, bicompleteness and the formulas for \( EA(X) \), \( d^0A(X) \) and \( d^1A(X) \) characterize the functor \( A \).

We have the following restrictions of \( A \) to full subcategories:

\[
\begin{align*}
  A : \mathcal{C}(\mathcal{A}) & \rightarrow \mathcal{F}(\mathcal{A}), & A : \mathcal{C}_{+}(\mathcal{A}) & \rightarrow \mathcal{F}_{+}(\mathcal{A}), \\
  A : \mathcal{C}_{-}(\mathcal{A}) & \rightarrow \mathcal{F}_{-}(\mathcal{A}), & A : \mathcal{C}_{-}(\mathcal{A}) & \rightarrow \mathcal{F}_{-}(\mathcal{A}).
\end{align*}
\]

3.5. Definition. For two morphisms \( f, g : X \rightarrow Y \) in \( \mathcal{F}(\mathcal{A}) \), a homotopy \( D \) between \( f \) and \( g \) is a homotopy \( D \) between \( F(f) \) and \( F(g) \) such that \( D(F_pX) \subseteq F_{p+1}Y \) for each \( p \). Moreover, \( D \) is a fibre homotopy if \( D(F_pX) \subseteq F_pY \) for each integer \( p \).

3.6. Remarks. A homotopy of the first kind in \( \mathcal{C}(\mathcal{A}) \) induces a homotopy in \( \mathcal{F}(\mathcal{A}) \) via the assembly functor \( A \). A homotopy of the second kind in \( \mathcal{C}(\mathcal{A}) \) induces a fibre homotopy in \( \mathcal{F}(\mathcal{A}) \) via the assembly functor \( A \).

Let \( \mathcal{F}(\mathcal{A})_\pi \) denote the quotient of \( \mathcal{F}(\mathcal{A}) \) by the homotopy relation and \( \mathcal{F}(\mathcal{A})_{F\pi} \) the quotient by the fibre homotopy relation. Note that \( \mathcal{F}(\mathcal{A})_\pi \) is also a quotient of \( \mathcal{F}(\mathcal{A})_{F\pi} \). Hence \( A \) induces quotient functors, which are defined as

\[
\begin{align*}
  A_1 : \mathcal{C}(\mathcal{A})_\pi & \rightarrow \mathcal{F}(\mathcal{A})_\pi, & A_1 : \mathcal{C}(\mathcal{A})_{F\pi} & \rightarrow \mathcal{F}(\mathcal{A})_{F\pi}. \\
  A_1 : \mathcal{C}(\mathcal{A})_\pi & \rightarrow \mathcal{F}(\mathcal{A})_\pi.
\end{align*}
\]

Moreover,

\[
A_1 \circ : \mathcal{C}(\mathcal{A})_\pi \rightarrow \mathcal{F}(\mathcal{A})_{F\pi}.
\]

Finally note that the functor \( F_\pi : \mathcal{F}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}) \) induces functors, also denoted \( F_\pi \), which are defined as

\[
\begin{align*}
  F_\pi : \mathcal{F}(\mathcal{A})_\pi & \rightarrow \mathcal{C}(\mathcal{A})_\pi, & F_\pi : \mathcal{F}(\mathcal{A})_{F\pi} & \rightarrow \mathcal{C}(\mathcal{A})_{F\pi}.
\end{align*}
\]

The two quotients \( A_1 \) and \( A_{11} \) restrict to various full subcategories described by positivity conditions.

3.7. Remarks. Let \( SS_s(\mathcal{A}) \) denote the category of spectral sequences \( E^r \) over \( \mathcal{A} \) defined for \( s < r \). Then the usual construction yields functors

\[
\begin{align*}
  E_\pi : \mathcal{F}(\mathcal{A}) & \rightarrow SS_0(\mathcal{A}), \\
  E^r : \mathcal{F}(\mathcal{A})_{F\pi} & \rightarrow SS_1(\mathcal{A}), \\
  E_\pi : \mathcal{F}(\mathcal{A})_\pi & \rightarrow SS_2(\mathcal{A}).
\end{align*}
\]

4. Generalities on differential derived functors

In this section, \( \mathcal{A} \) will denote a bisummable abelian category. Let \( \text{Gr}(\mathcal{A}) \) denote the category of graded objects over \( \mathcal{A} \), i.e. the full subcategory of \( \mathcal{C}(\mathcal{A}) \) generated
by the objects with zero differential. The homology functor $H$ is defined as

$$H : \mathcal{C}(\mathcal{A}) \to \text{Gr}(\mathcal{A}) \quad \text{or} \quad H : \mathcal{C}(\mathcal{A})_\pi \to \text{Gr}(\mathcal{A}),$$

and is composed with $F_\infty$ to yield

$$HF_\infty : \mathcal{F}\mathcal{C}(\mathcal{A}) \to \text{Gr}(\mathcal{A}),$$

$$HF_\infty : \mathcal{F}\mathcal{C}(\mathcal{A})_\pi \to \text{Gr}(\mathcal{A}),$$

$$HF_\infty : \mathcal{F}\mathcal{C}(\mathcal{A})_{F_\pi} \to \text{Gr}(\mathcal{A}).$$

Next consider a functor $T$ of $n$ variables,

$$T : \mathcal{X}_1 \times \ldots \times \mathcal{X}_n \to \mathcal{E}(\mathcal{A}),$$

where $\mathcal{X}_i$ is an additive category with injective class $\mathcal{Q}_i$. For $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ the functor

$$T_r = T_\mathcal{Q} : \mathcal{X}_1 \times \ldots \times \mathcal{X}_n \to \mathcal{E}(\mathcal{C}(\mathcal{A}))_\pi$$

is defined as in 2.3, where $r$ is an injective resolution

$$r : \mathcal{X}_1 \times \ldots \times \mathcal{X}_n \to \mathcal{E}(\mathcal{X}_1)_\pi \times \ldots \times \mathcal{E}(\mathcal{X}_n)_\pi.$$

4.1. Definition. The differential derived functor of $T$ of the first kind, denoted $R_{\mathcal{Q},1}T$, is

$$HF_\infty A_{\mathcal{Q}} T_\mathcal{Q} : \mathcal{X}_1 \times \ldots \times \mathcal{X}_n \to \text{Gr}(\mathcal{A})$$

and the differential derived functor of $T$ of the second kind, denoted $R_{\mathcal{Q},II}T$, is

$$HF_\infty (A_{\mathcal{Q}} 0) T_\mathcal{Q} : \mathcal{X}_1 \times \ldots \times \mathcal{X}_n \to \text{Gr}(\mathcal{A}).$$

The functor $R_{\mathcal{Q},1}T$ is the abutment of the spectral sequence of the functor

$$A_{\mathcal{Q}} T_\mathcal{Q} : \mathcal{X} \to \mathcal{F}\mathcal{C}(\mathcal{A})_\pi$$

denoted $E' R_{\mathcal{Q},1}T$, and $R_{\mathcal{Q},II}T$ is the abutment of the spectral sequence of the functor

$$A_{\mathcal{Q}} 0 T_\mathcal{Q} : \mathcal{X} \to \mathcal{F}\mathcal{C}(\mathcal{A})_{F_\pi},$$

denoted $E' R_{\mathcal{Q},II}T$. From 3.7 it follows that $E' R_{\mathcal{Q},1}T$ is defined for $r \geq 2$ and $E' R_{\mathcal{Q},II}T$ for $r \geq 1$.

In [19], differential derived functors were all of the first kind.

4.2. Definition. A functor

$$T : \mathcal{X}_1 \times \ldots \times \mathcal{X}_n \to \mathcal{E}(\mathcal{A})$$

is well balanced relative to the injective class $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ if whenever the $I_j$ are in $\mathcal{Q}_j$ and $X' \to X \to X''$ is in $\text{Seq}(\mathcal{Q}_r)$ for any $r$, $1 \leq r \leq n$, then
(1) the sequence
\[ HT(I_1, \ldots, X', \ldots, I_n) \Rightarrow HT(I_1, \ldots, I_{r-1}, X, I_{r+1}, \ldots, I_n) \Rightarrow HT(I_1, \ldots, X'', \ldots, I_n) \]
is exact in \( \text{Gr}(\mathcal{A}) \), and

(2) the sequence
\[ T(I_1, \ldots, X', \ldots, I_n) \Rightarrow T(I_1, \ldots, I_{r-1}, X, I_{r+1}, \ldots, I_n) \Rightarrow T(I_1, \ldots, X'', \ldots, I_n) \]
is exact in \( C(\mathcal{A}) \).

Condition (1) is the definition of balanced in [19] and is what is needed to show that differential derived functors of the first kind can be calculated by resolving in \( n-1 \) variables, for \( n > 1 \). Condition (2) is what is needed to show that differential derived functors of the second kind can be calculated by resolving in only \( n-1 \) variables, for \( n > 1 \).

Hence the natural morphism from the coproduct to the product (with appropriate changes in sign) induces a morphism of functors \( \tau : F_\infty A\sigma \Rightarrow F_\infty A \) from \( C_-(\mathcal{A}) \) to \( C(\mathcal{A}) \).

4.4. Definition. An object \( X \) in \( C_-(\mathcal{A}) \) is tapered if
\[ H\tau(X) : HF_\infty A\sigma(X) \Rightarrow HF_\infty A(X) \]
is an isomorphism.

If \( X \) is tapered and \( X' \) is isomorphic with \( X \) in \( C_-(\mathcal{C}(\mathcal{A}))_n \), then \( X' \) is tapered.

4.4. Definition. A functor
\[ T : X_1 \times \ldots \times X_n \to C(\mathcal{A}) \]
is tapered relative to the injective class \( \mathcal{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_n) \) if
\[ T_{\mathcal{I}} : X_1 \times \ldots \times X_n \to C_-(\mathcal{C}(\mathcal{A}))_n \]
takes values in the full subcategory generated by the tapered objects.

When \( T \) is tapered relative to \( \mathcal{I} \), the natural morphism \( R_{\mathcal{I}} T \Rightarrow R_{\mathcal{I}} T \) induced by \( \tau \) is an isomorphism. One has two different spectral sequences abutting to the same functor. Note that if \( T \) takes values in \( C_-(\mathcal{A}) \), then it is tapered relative to any injective class.

Tapered functors were considered in [15], where it was proved that \( \text{Cotor}^C \) is tapered when \( C \) is simply connected. This was done using tapered resolutions.

The dual of the preceding situation affords the definition of differential derived functors of the first and second kind relative to projective classes.
5. Remarks on the differential Cotor functor

For a right \( C \)-comodule \( L \) and a left \( C \)-comodule \( M \) the exact sequence

\[
0 \to L \otimes_C M \to L \otimes M \xrightarrow{f} L \otimes C \otimes M,
\]

where \( f = \Delta(L) \otimes M - L \otimes \Delta(M) \), defines the cotensor product \( L \otimes_C M \) and the functor \( (\cdot \otimes_C \cdot) \) : \( \text{Co}^\infty C \times \text{Co}^\infty M \to \mathcal{E}_+(R) \).

5.1. Remark. The cotensor product functor is well balanced with respect to the standard injective classes. Hence for \( \text{Cotor}^C \) and \( \text{Cotor}^{C,\Pi} \), the first and second derived functors of \( (\cdot \otimes_C \cdot) \), one can calculate by resolving just one variable.

For a right \( C \)-comodule \( L \) and a left \( C \)-comodule \( M \) we denote by \( L^\# \) the right \( C^\# \)-comodule and \( M^\# \) the left \( C^\# \)-comodule, where \( C^\# \) is \( C \) with \( d = 0 \), \( L^\# \) is \( L \) with \( d = 0 \), and \( M^\# \) is \( M \) with \( d = 0 \). Then \( L^\# \otimes_C M^\# \) is in \( \text{Gr}(R) \) and the ordinary derived functor \( \text{Cotor}^{C,\Pi}(L^\#, M^\#) \) is in \( \text{Gr}(\text{Gr}(R)) \).

5.2. Remark. The \( E^1 \)-term of the spectral sequence associated with \( \text{Cotor}^{C,\Pi}(L, M) \) is given by

\[
E^1 \text{Cotor}^{C,\Pi}(L, M) = \text{Cotor}^{C^\#,\Pi}(X^\#, Y^\#).
\]

6. Spectral properties of the cotensor product of skeletally filtered comodules

For \( L \) in \( \mathcal{F} \text{Co}^\infty C \) and \( M \) in \( \mathcal{F} \text{Co}^\infty M \) the exact sequence

\[
0 \to L \otimes_C M \to L \otimes M \xrightarrow{f} L \otimes C \otimes M,
\]

where \( f = \Delta(L) \otimes M - L \otimes \Delta(M) \), defines the cotensor product \( L \otimes_C M \) and the functor

\[
(\cdot \otimes_C \cdot) : \mathcal{F} \text{Co}^\infty C \times \mathcal{F} \text{Co}^\infty M \to \mathcal{F}^R \mathcal{E}_+(R),
\]

which is \( R \)-split left filtering exact on \( R \)-split filtering exact sequences.

Now define

\[
T : \mathcal{E}_- X \times \mathcal{E}_- Y \to \mathcal{F}^R \mathcal{E}(R)
\]

for \( X = \mathcal{F} \text{Co}^\infty C \) and \( Y = \mathcal{F} \text{Co}^\infty M \) as a filtered object

\[
T(X, Y) = \prod_{p,q} \mathcal{P}^{p,q}(X_p \otimes_C Y_q)
\]

with differential \( d = d' + d'' \) such that

\[
F_- T(X, Y) = T(E_- F_- X, E_- F_- Y)
\]

and

\[
d^1 F_p T(X, Y) \subset F_{p-1} T(X, Y).
\]
6.1. **Definition.** The **skeletal spectral functor** on the category $\mathcal{F} \text{CoM}_C \times \mathcal{F} \text{CoM}_C$ to the category of spectral sequences of R-modules with abutment is defined by the relations

$$E'(L, M) = E'T(U_*(L), U_*(M)), \quad E'(f, g) = E'T(U_*(f), U_*(g)),$$

where $U_*$ is the canonical resolution, and for the abutments,

$$\text{Cotor}^C(F_\infty X, F_\infty Y) = HF_\infty T(U_*(L), U_*(M)),
\text{Cotor}^C(F_\infty f, F_\infty g) = HF_\infty T(U_*(f), U_*(g)).$$

This is an additive functor of two variables.

Observe that

$$E^1_{*, q}(\cdot \subseteq C \cdot) : \mathcal{F} \text{CoM}_C \times \mathcal{F} \text{CoM}_C \to \mathcal{C}(R)$$

is an additive functor. In the next proposition we relate it to the skeletal spectral functor.

6.2. **Proposition.** The functor

$$E^1_{*, q}(\cdot \subseteq C \cdot) : \mathcal{F} \text{CoM}_C \times \mathcal{F} \text{CoM}_C \to \mathcal{C}(R)$$

is balanced with respect to the standard injective classes, and its differential derived functor is the component $E^2_{*, q}$ of the skeletal functor for each integer $q$.

**Proof.** To show that the functor is balanced (see [19, (6.5)]) we observe that for an injective $L$ and $0 \to M' \to M \to M'' \to 0$ a short sequence in the injective class, the sequence

$$0 \to L \subseteq C \cdot M' \to L \subseteq C \cdot M \to L \subseteq C \cdot M'' \to 0$$

is a split exact sequence in $\mathcal{F} \mathcal{C}(R)$. This implies that

$$0 \to E^1_{*, q}(L \subseteq C \cdot M') \to E^1_{*, q}(L \subseteq C \cdot M) \to E^1_{*, q}(L \subseteq C \cdot M'') \to 0$$

is a split exact sequence in $\mathcal{C}(R)$. A similar assertion holds for the variables reversed.

For the second statement, note that if $T(q)$ is the standard assembly functor, as just before 6.1, for the functor $E^1_{*, q}(L \subseteq C \cdot M)$, then

$$T(q)(U_*(L), U_*(M)) = E^1_{*, q}T(U_*(L), U_*(M))$$

for objects and morphisms. This proves the proposition. \(\square\)

6.3. **Corollary.** The skeletal spectral functor on the category $\mathcal{F} \text{CoM}_C \times \mathcal{F} \text{CoM}_C$ is balanced with respect to the standard injective classes.

Note that the corollary implies that either morphism in the diagram

$$T(L, U_*(M)) \to T(U_*(L), U_*(M)) \leftarrow T(U_*(L), M)$$
induces an isomorphism of spectral sequences with abutment, and further that either $U_p(L)$ or $U_p(M)$ could be replaced by arbitrary injective resolutions $L \rightarrow X_*$ and $M \rightarrow Y_*$ of $L$ and $M$.

6.4. Definition. A pair $(L, M)$ of graded $R$-modules is relatively flat if $\text{Tor}_q^R(L, M) = 0$ for $q > 0$.

6.5. Proposition. If $C$ is an $R$-flat coalgebra, and $(L, M)$ is in $\mathcal{F} \text{CoM} \times \mathcal{F} \text{CoM}$ such that $(E^1(L), E^1(M))$ is relatively flat over $R$, then for the skeletal spectral sequence for $(L, M)$ it follows that

$$E^2(L, M) = \text{Cotor}^C(E^1(L), E^1(M)).$$

Proof. For an injective resolution

$$0 \rightarrow M \rightarrow Y_0 \rightarrow Y_{-1} \rightarrow \ldots,$$

we have an injective resolution

$$0 \rightarrow E^1(M) \rightarrow E^1(Y_0) \rightarrow E^1(Y_{-1}) \rightarrow \ldots.$$

To get $E^1(L, M)$, we consider

$$E^0(L \odot_C X_0) \rightarrow E^0(L \odot_C X_{-1}) \rightarrow \ldots,$$

and then assemble to get a complex whose homology is $E^1(L, M)$. Since $(E^1(L), E^1(M))$ is relatively flat, the pair $(E^1(L), E^1(Y_j))$ is relatively flat, and hence $E^1(L \odot_C Y_j) = E^1(L) \odot_C E^1(Y_j)$. Assemble

$$E^1(L) \odot_C E^1(Y_0) \rightarrow E^1(L) \odot_C E^1(Y_{-1}) \rightarrow \ldots,$$

and take homology getting $\text{Cotor}^C(E^1(L), E^1(M))$; but this sequence is the same as

$$E^1(L \odot_C Y_0) \rightarrow E^1(L \odot_C Y_{-1}) \rightarrow \ldots,$$

which assembled has homology $E^2(L, M)$. This proves the proposition. $\Box$

6.6. Corollary. With the notation and hypotheses of 6.5, if in addition $E^1(M) = C \odot H$ is an extended comodule with $d(H) = 0$, then $E^2(L, M) = H(E^1(L) \odot H)$.

This corollary will be used to identify the differential Cotor with the homology of appropriate induced fibrations; see III.5.6.
Let $\text{Alg}(R)$ denote the category of supplemented differential $R$-algebras, and let $\text{Coalg}(R)$ denote the category of connected differential $R$-coalgebras. In this part, algebra will mean object of $\text{Alg}(R)$, and coalgebra will mean object of $\text{Coalg}(R)$, although for some considerations the connectedness hypothesis will not be needed. The main purpose of this part is to define the classifying coalgebra functor

$$B : \text{Alg}(R) \to \text{Coalg}(R)$$

and the loop algebra functor

$$\Omega : \text{Coalg}(R) \to \text{Alg}(R),$$

prove the adjoint relation $\Omega \dashv B$, and study the homotopy properties of the adjunction morphisms $\beta : C \to B\Omega(C)$ and $\alpha : \Omega B(A) \to A$.

I. Twisting morphisms

For morphisms of graded modules $f : X \to X'$ of degree $m$ and $g : Y \to Y'$ of degree $n$, recall that

$$f \circ g : X \otimes Y \to X' \otimes Y'$$

is the morphism of degree $m + n$ such that

$$(f \circ g)_{p+q}(x \otimes y) = (-1)^{pq} f_p(x) \otimes g_q(y)$$

for $x \in X_p$, $y \in Y_q$. Moreover,

$$(f \circ Y')(X \circ g) = f \circ g = (-1)^{mn} (X' \circ g)(f \circ Y)'$$

1.1. Definition. Let $A$ be an algebra and $C$ a coalgebra. For morphisms of the underlying graded $R$-module structure $f : C \to A$ of degree $m$ and $g : C \to A$ of degree $n$, the cup product $f \cup g : C \to A$ of $f$ and $g$ is the morphism of degree $m + n$ which is the composite

$$C \xrightarrow{\Delta(C)} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\phi(A)} A$$

The cup product is associative, that is, if in addition $h : C \to A$ is of degree $r$, then

$$(f \cup g) \cup h = f \cup (g \cup h),$$

and $\epsilon(A) \eta(C)$ is a unit for the cup product operation. Further, if $A$ and $C$ are commutative, then so is the cup product, i.e.

$$f \otimes g = (-1)^{mn} g \otimes f.$$
For a morphism \( f : C \to A \) of degree \( m \) of graded \( R \)-modules, we denote by
\[
d_f : A \otimes C \to A \otimes C
\]
the composite
\[
A \otimes C \xrightarrow{\Delta(C)} A \otimes C \otimes C \xrightarrow{A \otimes f \otimes C} A \otimes A \otimes C \xrightarrow{\otimes(A) \otimes C} A \otimes C
\]
and by
\[
d_f : C \otimes A \to C \otimes A
\]
the composite
\[
C \otimes A \xrightarrow{\Delta(C) \otimes A} C \otimes C \otimes A \xrightarrow{C \otimes f \otimes A} C \otimes A \otimes A \xrightarrow{C \otimes \otimes(A)} C \otimes A.
\]
In both cases \( d_f \) is a morphism of graded \( R \)-modules of degree \( m \).

1.2. Proposition. Let \( A \) be an algebra, let \( C \) be a coalgebra, and let \( f : C \to A \) be a morphism of degree \( -1 \) of the underlying graded \( R \)-modules. Then \( d_f : A \otimes C \to A \otimes C \) has the property that \( d_f^2 = -d_f \cup f \) and \( d_f : C \otimes A \to C \otimes A \) has the property that \( d_f^2 = d_f \cup f \).

The proof of the proposition is routine and left to the reader. □

The difference in sign comes from the relation
\[
(f \otimes C)(A \otimes f) = f \otimes f = -(C \otimes f)(f \otimes A).
\]

1.3. Remark. Let \( X \) be a differential graded \( R \)-module and \( d'' : X \to X \) a morphism of degree \( -1 \) of the underlying graded structures. Then \( d''(X) \) is a differential on the graded \( R \)-module \( X \) (i.e. has square zero) if and only if \( d'' d'' = -Dd'' \), and \( d'(X) - d'' \) is a differential if and only if \( d'' d'' = Dd'' \), where \( D : [X, X]_{-1} \to [X, X]_{-2} \) is the differential. This is immediate from the relation \( Dd'' = d(X)d''d'' + d''d(X) \).

1.4. Definition. Let \( A \) be an algebra, and let \( C \) be a coalgebra. A twisting morphism \( \tau : C \to A \) is a morphism of degree \( -1 \) of the underlying graded modules such that
\[
D\tau = \tau \cup \tau, \quad \tau\eta(C) = 0 = e(A) \tau.
\]

Let \( \tau : C \to A \) be a twisting morphism. The differential \( R \)-module \( A \otimes_\tau C \) is the graded \( A \otimes C \) with the differential
\[
d(A \otimes_\tau C) = d(A \otimes C) + d_\tau,
\]
and \( C \otimes_\tau A \) is the graded \( C \otimes A \) with the differential
\[
d(C \otimes_\tau A) = d(C \otimes A) - d_\tau.
\]
These are differentials by 1.2 and 1.3. In addition, $A \otimes \tau C$ is a left $A$-module, right $C$-comodule, and $C \otimes \tau A$ is a left $C$-comodule, right $A$-module, which, neglecting differentials, are each extended. For a morphism $f : C' \rightarrow C$ of coalgebras, $g : A \rightarrow A'$ of algebras, and a twisting morphism $\tau$, we have natural isomorphisms

$$A \otimes_{\tau} C' = (A \otimes C) \circ_{C} C', \quad A' \otimes_{\gamma} C = A' \otimes_{A} (A \otimes C),$$

$$C' \otimes_{\tau} A = C' \circ_{C} (C \otimes_{\tau} A), \quad C \otimes_{\gamma} A' = (C \otimes_{\tau} A) \circ_{A} A'.$$

These constructions lead to the following more general concepts.

1.5. Definition. Let $A$ be an algebra and $C$ a coalgebra. An $(A, C)$-bimodule $M$ is a positive differential $R$-module $M$ together with morphisms

$$\phi(M) : A \otimes M \rightarrow M, \quad \Delta(M) : M \rightarrow M \otimes C$$

such that:

1. $(M, \phi(M))$ is a left $A$-module, and $\Delta(M)$ is a morphism of left $A$-modules, where $\phi(M \otimes C) = \phi(M) \otimes C$ is the left $A$-module structure morphism on $M \otimes C$, and

2. $(M, \Delta(M))$ is a right $C$-comodule, and $\phi(M)$ is a morphism of right $C$-comodules, where $\Delta(A \otimes M) = A \otimes \Delta(M)$ is the right $C$-comodule structure morphism on $A \otimes M$.

An $(C, A)$-bimodule $N$ is a positive differential $R$-module $M$ together with morphisms

$$\phi(N) : N \otimes A \rightarrow N, \quad \Delta(N) : N \rightarrow C \otimes N$$

such that:

1. $(N, \phi(N))$ is a right $A$-module, and $\Delta(N)$ is a morphism of right $A$-modules, where $\phi(N \otimes C) = C \otimes \phi(N)$, and

2. $(N, \Delta(N))$ is a left $C$-comodule, and $\phi(N)$ is a morphism of left $C$-comodules, where $\Delta(N \otimes C) = \Delta(N) \otimes C$.

A morphism $f : M' \rightarrow M''$ of bimodules is a morphism of differential $R$-modules which is both an $A$-module and $C$-comodule morphism. Composition being induced by composition in $C(R)$, the categories of $(A, C)$-bimodules and $(C, A)$-bimodules are defined. These are additive categories with cokernels, and they are abelian if $C$ is flat over $R$.

1.6. Definition. Let $A$ be an algebra and $C$ a coalgebra. An $(A, C)$-bundle $M$ is an $(A, C)$-bimodule $M$ such that, neglecting differentials, there exists an isomorphism $\lambda : M \rightarrow A \otimes C$ of left $A$-modules and right $C$-comodules.

An $(C, A)$-bundle $N$ is a $(C, A)$-bimodule $N$ such that, neglecting differentials, there exists an isomorphism $\lambda : N \rightarrow C \otimes A$ of right $A$-modules and left $C$-comodules.

The module and comodule structures on $A \otimes C$ and $C \otimes A$ are the extended ones. A choice of $\lambda$ will be called a choice of coordinates for the bundle.
1.7. Remark. The bimodules $A \otimes C$ and $C \otimes A$, where $\tau : C \to A$ is a twisting morphism, are examples of bundles. A theorem of Gugenheim [16] asserts that given any $(A, C)$-bundle $M$ there exists a twisting morphism $\tau : C \to A$ and an isomorphism $A \otimes C \to M$.

Let $M$ be a $(C, A)$-bimodule. Tensoring $M$ over $A$ with $\varepsilon(A) : A \to R$ and cotensoring $M$ over $C$ with $\eta(C) : R \to C$, we have the diagram

$$
\begin{array}{ccc}
R \otimes_C M & \xrightarrow{i} & M \\
\downarrow{\pi} & & \\
M \otimes_A R & & 
\end{array}
$$

Here the left $C$-comodule $M \otimes_A R$ is called the base comodule and $\pi$ is an epimorphism of left $C$-comodules. Moreover, the right $A$-module $R \otimes_C M$ is called the fibre module and $i$ is a monomorphism of right $A$-modules. In the case where $M$ is a $(C, A)$-bundle, the base is canonically isomorphic to $C$ as a $C$-comodule and the fibre to $A$ as an $A$-module. Similar definitions apply to $(A, C)$-bimodules.

2. Derivations on algebras and coalgebras

2.1. Definition. Let $A$ be an algebra and $m$ an integer. A derivation $\delta : A \to A$ of degree $m$ is a morphism of the underlying graded structures of degree $m$ such that

1. $\varepsilon(A) \delta = 0$,
2. $D\delta = 0$,
3. the diagram

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\phi(A)} & A \\
\delta \otimes A + A \otimes \delta & \downarrow{\delta} & \downarrow{\delta} \\
A \otimes A & \xrightarrow{\phi(A)} & A 
\end{array}
$$

of graded modules and generalized morphisms is commutative.

A differential $\delta : A \to A$ is a derivation of degree $-1$ such that $\delta^2 = 0$.

Observe that condition (2) says that $\delta$ is a morphism of degree $m$ of differential $R$-modules. Thus the diagram of 2.1(3) may be viewed as a commutative diagram of differential $R$-modules and generalized morphisms. These remarks apply also to the next definition.

2.2. Definition. Let $C$ be a coalgebra and $m$ an integer. A derivation $\delta : C \to C$ of degree $m$ is a morphism of the underlying graded structures of degree $m$ such that
(1) $\delta \eta(C) = 0$,
(2) $D\delta = 0$,
(3) the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta(C)} & C \otimes C \\
\downarrow \delta & & \downarrow \delta \otimes C + C \otimes \delta \\
C & \xrightarrow{\Delta(C)} & C \otimes C
\end{array}
$$

of graded modules and generalized morphisms is commutative.

A differential $\delta : C \to C$ is a derivative of degree $-1$ such that $\delta^2 = 0$.

Now let $T : \mathcal{C}_+(R) \to \text{Alg}(R)$ denote the tensor-algebra functor, where for a differential $R$-module $X$ the underlying differential $R$-module of $T(X)$ is $\bigoplus_{0 \leq n} X^n \otimes R$. Here $X^n \otimes R$ is the $n$-fold tensor product of $X$ with itself, and the multiplication on $T(X)$ is induced by the natural isomorphisms

$$X^n \otimes R \to X^{(m+n)\otimes}.$$

Noting that $T(0) = R$, we define the supplementation of $T(X)$ as $\epsilon(T(X)) = T(0)$, where $0 : X \to 0$.

For any algebra $A$, let $I(A)$ denote its augmentation ideal viewed as an object of $\mathcal{C}_+(R)$, where $I(A)$ is the kernel of $\epsilon(A) : A \to R$. Observe that $I : \text{Alg}(R) \to \mathcal{C}_+(R)$ is a functor. Further note that

$$IT(X) = \bigsqcup_{1 \leq n} X^n \otimes R,$$

and let

$$\beta(X) : X = X^1 \otimes R \to IT(X)$$

be the canonical injection for $X$ in $\mathcal{C}_+(R)$.

2.3. Remark. For an algebra $A$ there is a morphism $\alpha(A) : T(I(A)) \to A$ such that

$$I(\alpha(A)) \beta(I(A)) = I(A).$$

and $(\alpha, \beta) : T \to I$ is an adjoint pair of functors with coadjoint the tensor algebra functor $T$ and adjoint $I$. This statement is just an expression of the usual universal property of $\beta(X) : X \to IT(X)$.

2.4. Proposition. Let $X$ be a differential $R$-module. For each $\theta \in Z[X, I(T(X))]_m$ there is a unique derivation $\delta_\theta : T(X) \to T(X)$ of degree $m$ such that

$$I(\delta_\theta) \beta(X) = \theta.$$
Moreover, if \( \delta : T(X) \to T(X) \) is a derivation of odd degree \( m \), then \( \delta^2 \) is the derivative of degree \( 2m \) induced by the element \( \Omega(\delta^2, \beta(X)) \) of \( Z \{X, \Omega(T(X))\}_{2m} \).

The proof of the proposition results from a simple extension of \( \theta \) on \( X \) to \( \delta_\theta \) on \( T(X) \) as in proving the adjoint relation \( T \to I \), and is left to reader. It effectively determines the derivation \( T(X) \to T(X) \), and for \( \delta \) of odd degree it furnishes a criterion for seeing if \( \delta^2 = 0 \).

Now let \( \mathcal{C}_c(R) \) denote the full subcategory of \( \mathcal{C}_+(R) \) determined by connected differential \( R \)-modules \( X \), i.e. those \( X \) with \( X_0 = 0 \). Observe that

\[
(X^{n\otimes})_q = 0
\]

for \( q < n \). Let

\[
T' : \mathcal{C}_c(R) \to \text{Coalg}(R)
\]

denote the tensor coalgebra functor, where for a differential \( R \)-module \( X \) the underlying differential \( R \)-module of \( T'(X) \) is

\[
\bigoplus_{0 \leq n} X^{n\otimes} = \prod_{0 \leq n} X^{n\otimes}
\]

because \( X_0 = 0 \). The comultiplication on \( T'(X) \) is induced by the natural isomorphisms

\[
x^{(m+n)\otimes} \cong x^{m\otimes} \otimes x^{n\otimes}.
\]

Noting that \( T'(0) = R \), we define the supplementation of \( T'(X) \) as

\[
\eta(T'(X)) = T'(0),
\]

where \( 0 : 0 \to X \).

For any coalgebra \( C \), let \( I(C) \) denote its augmentation coideal viewed as an object of \( \mathcal{C}_c(R) \), where \( I(C) \) is the cokernel of \( \eta(C) : R \to C \). Observe that \( I : \text{Coalg}(R) \to \mathcal{C}_c(R) \) is a functor. Further note that

\[
IT'(X) = \bigoplus_{1 \leq n} X^{n\otimes} = \prod_{1 \leq n} X^{n\otimes},
\]

and let \( \alpha(X) : IT'(X) \to X = X^{1\otimes} \) be the canonical projection for \( X \) in \( \mathcal{C}_c(X) \).

**2.5. Remark.** For a coalgebra \( C \) there is a morphism \( \beta(C) : C \to T'I(C) \) such that

\[
\beta(I(C)) I(\beta(C)) = I(C),
\]

and \( (\alpha, \beta) : I \to T' \) is an adjoint pair of functors with adjoint the tensor coalgebra functor \( T' \) and coadjoint \( I \). This statement is just an expression of the usual universal property of \( \alpha : IT'(X) \to X \).
2.6. Proposition. Let $X$ be a connected differential $R$-module. For each \( \theta \in \mathbb{Z}[\text{IT}'(X), X]_m \) there is a unique derivation \( \delta_\theta : \text{T}'(X) \to \text{T}'(X) \) of degree $m$ such that \( \alpha(X) I(\delta_\theta) = \theta \). Moreover, if \( \delta : \text{T}'(X) \to \text{T}'(X) \) is a derivation of odd degree $m$, then $\delta^2$ is the derivation of degree $2m$ induced by the element $\alpha(X) I(\delta^2)$ of $\mathbb{Z}[\text{IT}'(X), X]_{2m}$.

3. The classifying construction and loop construction

For a differential $R$-module $X$ there is a natural isomorphism

\[
(s'(X))^{n\otimes} = s'^{n}(X^{n\otimes}).
\]

Also there is a canonical isomorphism $\sigma'(X) : X \to s'(X)$ of degree $r$ such that for any differential $R$-module $Y$ both

\[
[s'(X), Y] : [s'(X), Y] \to [X, Y]
\]

and

\[
[X, \sigma'(Y)] : [X, Y] \to [X, s'(Y)]
\]

are differential isomorphisms of degree $r$. Thus we have canonical isomorphisms

\[
s'[s'(X), Y] = [X, Y] = s'^{-r}[X, s'(Y)].
\]

Clearly $\mathcal{C}_+(R)$ and $\mathcal{C}_c(R)$ are stable under $s'$ for $r \geq 0$, and for $X$ in $\mathcal{C}_c(R)$, $s^{-1}(X)$ is in $\mathcal{C}_+(R)$.

For an algebra $A$ consider the commutative diagram

\[
\begin{array}{ccc}
IT'(sI(A)) & \xrightarrow{p} & (sI(A))^2 \\
\downarrow{\theta(A)} & & \downarrow{\theta(A)} \\
sI(A) & & sI(A)
\end{array}
\]

where $\theta(A) : (sI(A))^{2\oplus} \to sI(A)$ is the differential morphism of degree $-1$ induced by the multiplication on $I(A)$ and $p$ is the natural projection. We denote by

\[
\delta(A) : \text{T}'(sI(A)) \to \text{T}'(sI(A))
\]

the derivation of degree $-1$ induced by $\bar{\theta}(A)$ using 2.6.

For a coalgebra $C$ consider the commutative diagram

\[
\begin{array}{ccc}
s^{-1} I(C) & \xrightarrow{q} & IT(s^{-1} I(C)) \\
\downarrow{\theta(C)} & & \downarrow{\theta(C)} \\
(s^{-1} I(C))^2 & & \end{array}
\]
where

$$\theta(C) : s^{-1} I(C) \to (s^{-1} I(C))^2$$

is the differential morphism of degree $-1$ induced by the comultiplication on $I(C)$ and $q$ is the natural injection. We denote by

$$\delta(C) : T(s^{-1} I(C)) \to T(s^{-1} I(C))$$

the derivation of degree $-1$ induced by $\theta(C)$ using 2.4.

3.1. Lemma. (1) For an algebra $A$ the derivation

$$\delta(A) : T'(sl(A)) \to T'(sl(A))$$

is a differential, and for a morphism $f : A \to B$ of algebras,

$$T'(sl(f)) \delta(A) = \delta(B) T'(sl(f)).$$

(2) For a coalgebra $C$ the derivation

$$\delta(C) : T(s^{-1} I(C)) \to T(s^{-1} I(C))$$

is a differential, and for a morphism $f : C \to D$ of coalgebras,

$$T(s^{-1} I(f)) \delta(C) = \delta(D) T(s^{-1} I(f)).$$

Proof. The multiplication on $A$ is associative if and only if

$$\delta(A) (\theta(A) \otimes sl(A)) = -\theta(A) (sl(A) \otimes \theta(A)).$$

This is equivalent to saying

$$\alpha(sl(A)) J(\delta(C))^2 = 0.$$

Thus the first statement follows from 2.6. The relation coming from $f : A \to B$ is immediate. The last two statements are proved similarly using 2.4. \(\square\)

For a differential $\partial : A \to A$ on an algebra $A$ we denote by $A(\partial)$ the algebra which is $A$ as a graded algebra and

$$d(A(\partial)) = d(A) + \partial.$$

For a differential $\partial : C \to C$ on a coalgebra $C$ we denote by $C(\partial)$ the coalgebra which is $C$ as a graded coalgebra and

$$d(C(\partial)) = d(C) + \partial.$$

With this notation and Lemma 3.1 we can now make the main definitions of this section.
3.2. Definition. The classifying coalgebra functor

\[ B : \text{Alg}(R) \to \text{Coalg}(R) \]

assigns to each algebra \( A \) the coalgebra

\[ B(A) = T'(sI(A)) (\delta(A)) \]

and to each morphism \( f : A \to A' \) the morphism

\[ B(f) = T'(sI(f))(\delta). \]

Observe that \( B(f) : B(A) \to B(A') \) is a morphism of coalgebras by 3.1.

3.3. Definition. The loop algebra functor

\[ \Omega : \text{Coalg}(R) \to \text{Alg}(R) \]

assigns to each coalgebra \( C \) the algebra

\[ \Omega(C) = T(s^{-1} I(C)) (\delta(C)) \]

and to each morphism \( g : C \to C' \) the morphism

\[ \Omega(g) = T(s^{-1} I(g))(\delta). \]

Observe that \( \Omega(g) : \Omega(C) \to \Omega(C') \) is a morphism of algebras by 3.1.

3.4. Notation. For an algebra \( A \) let \( \tau(A) : B(A) \to A \) be the natural morphism of graded modules of degree \(-1\) which is the composite

\[ B(A) \to IB(A) \xrightarrow{\alpha(sI(A))} sI(A) \to I(A) \to A. \]

For a coalgebra \( C \) let \( \tau(C) : C \to \Omega(C) \) be the natural morphism of graded modules of degree \(-1\) which is the composite

\[ C \to I(C) \xrightarrow{s^{-1} I(C)} IS(C) \to \Omega(C). \]

If \( f : A' \to A'' \) is a morphism of algebras, then

\[ f \tau(A') = \tau(A'')B(f), \]

and if \( g : C' \to C'' \) is a morphism of coalgebras, then

\[ \Omega(f) \tau(C') = \tau(C'')f. \]

3.5. Proposition. (1) For an algebra \( A \) the morphism

\[ \tau(A) : B(A) \to A \]

is a twisting morphism such that for any twisting morphism \( \tau : C \to A \) there is a unique morphism \( f_\tau : C \to B(A) \) of coalgebras with \( \tau = \tau(A)f_\tau. \)
(2) For a coalgebra \( C \) the morphism
\[
\tau(C) : C \to \Omega(C)
\]
is a twisting morphism such that for any twisting morphism \( \tau : C \to A \) there is a unique morphism \( g_\tau : \Omega(C) \to A \) of algebras with \( \tau = g_\tau \tau(C) \).

Proof. That \( \tau(A) \) is a twisting morphism follows from the relations
\[
d(A) \tau(A) = -\tau(A) d(T'(sI(A))), \quad \tau(A) \cup \tau(A) = -\tau(A) \delta(A).
\]
For the universal mapping property let \( \tau' : I(C) \to I(A) \) be such that the composite
\[
C \to I(C) \to I(A) \to A
\]
is \( \tau \), and let \( f_\tau : C \to T'(sI(A)) \) be such that \( \alpha(sI(A)) I(f_\tau) \) is the composite
\[
I(C) \to I(A) \to sI(A).
\]
Hence we have a well-defined morphism \( f_\tau : C \to B(A) \) of graded coalgebras with \( \tau = \tau(A) f_\tau \), and it is unique with respect to this property. In addition, \( Df_\tau = 0 \) follows from the relation \( \tau = \tau(A) f_\tau \). Part (2) follows similarly. \( \square \)

For an algebra \( A \) let \( \alpha(A) : \Omega B(A) \to A \) be the morphism of algebras such that
\[
\alpha(A) \tau(B(A)) = \tau(A)
\]
given by 3.5(2), and for a coalgebra \( C \) let \( \beta(C) : C \to B \Omega(C) \) be the morphism of coalgebras such that
\[
\tau(\Omega(C)) \beta(C) = \tau(C)
\]
given by 3.5(1). These notations and the universal properties in 3.5 combine to yield the following theorem.

3.6. Theorem. The pair \((\alpha, \beta) : \Omega \to B\) is an adjoint pair of functors with coadjoint
\[
\Omega : \text{Coalg}(R) \to \text{Alg}(R)
\]
and adjoint
\[
B : \text{Alg}(R) \to \text{Coalg}(R).
\]

3.7. Remarks. Additively, the classifying coalgebra \( B(A) \) is the "bar construction" or, more exactly, the normalized bar construction of Eilenberg and MacLane [9, 12]. Additively, the loop algebra \( \Omega(C) \) is the "cobar construction" of Adams [1].

4. Homotopy properties of the adjunction morphisms

We will consider the adjunction morphism \( \alpha(A) : \Omega B(A) \to A \) by studying its kernel \( N(A) \to \Omega B(A) \) in the category of algebras. Since \( \alpha(A) \) is split surjective, we
can prove that it is a homotopy equivalence by showing the existence of a conical contract
(see IV, Section 1) \( \epsilon(A) \) on \( \Omega B(A) \), which in addition will be a functor in \( A \). Recall for this that

\[
0 \to \Omega B(A) \to \Omega B(A) / A \to 0
\]

is an exact sequence of differential modules split as graded objects. Similar considerations will hold for \( \beta(C) : C \to B \Omega(C) \).

4.1. Remark. Let \( j : C \to I(C) \) be the canonical retraction of \( C \) onto \( I(C) \) as a differential module. Let \( M = C \otimes Y \) be an extended left \( C \)-comodule, where

\[
\Delta(M) = \Delta(C) \otimes Y,
\]

and observe that

\[
r = C \otimes \epsilon(C) \otimes Y : C \otimes M \to M
\]

is a morphism of left \( C \)-comodules such that \( r \Delta(M) = M \). For \( i : I(C) \to C \), the kernel of \( \epsilon(C) : C \to R \), observe further that

\[
C \otimes j \otimes Y : C \otimes M \to C \otimes I(C) \otimes Y
\]

is the cokernel of \( \Delta(M) : M \to C \otimes M \), and

\[
C \otimes i \otimes Y : C \otimes I(C) \otimes Y \to C \otimes M
\]

is the kernel of \( r : C \otimes M \to M \).

We define inductively

\[
Y(0) = Y, \quad \text{and} \quad Y(n-1) = I(C) \otimes Y(n) \quad \text{for negative integers } n.
\]

For \( n \leq 0 \), let

\[
i(n) : C \otimes Y(n) \to C \otimes C \otimes Y(n)
\]

denote \( \Delta(C) \otimes Y(n) \), let

\[
\text{j}(n) : C \otimes C \otimes Y(n) \to C \otimes Y(n-1)
\]

denote \( C \otimes j \otimes Y(n) \), and let

\[
d(n) = i(n-1) j(n).
\]

The canonical resolution of \( C \otimes Y = M \) is

\[
C \otimes Y \to C \otimes C \otimes Y(0) \to \cdots
\]

Next let

\[
r(n) : C \otimes C \otimes Y(n) \to C \otimes Y(n)
\]

denote \( C \otimes \epsilon \otimes Y(n) \), let

\[
s(n) : C \otimes Y(n-1) \to C \otimes C \otimes Y(n)
\]

denote \( C \otimes i \otimes Y(n-1) \), and let
\[ c(n-1) = s(n-1) r(n-1) : C \otimes C \otimes Y(n-1) \to C \otimes C \otimes Y(\eta). \]

Now the canonical resolution of \( C \otimes Y = M \) is the direct sum of the identity \( C \otimes Y \to C \otimes Y \) with the resolution

\[ \mathbb{Q} \to C \otimes Y(-1) \xrightarrow{d(-1)} C \otimes C \otimes Y(-1) \xrightarrow{d(-1)} C \otimes C \otimes Y(-2) \to \ldots \]

of zero, and \( c \) is a homotopy of the latter resolution to zero such that \( c^2 = 0 \).

Note particularly that \( c(n) \) is a morphism of left \( C \)-comodules, and all the terms \( C \otimes C \otimes Y(-n) \) are naturally isomorphic to \( C \otimes (C^n)^{\otimes} \otimes M \).

4.2. Remark. If \( A \) is an algebra with \( \alpha(A)^2 = 0 \) and \( d(A) = 0 \), then \( B(A) = T'(sl(A)) \) as a coalgebra and \( dB(A) = 0 \). Moreover, for the tensor coalgebra \( T'(X) \) the co-multiplication

\[ \Delta T'(X) : T'(X) \to T'(X) \otimes T'(X) \]

restricts to an isomorphism \( IT'(X) \to T'(X) \otimes X \), and hence \( IT'(X) \) has canonically the structure of an extended \( T'(X) \)-comodule.

In order to study \( \Omega B(A) \) and \( N(A) \) in this case where \( \alpha(A)^2 = 0 \) and \( d(A) = 0 \), we divide the canonical resolution into two parts:

\[ 0 \to R \to T'(X) \to IT'(X) \to 0 \]

and the canonical resolution of \( IT'(X) \). The above considerations in 4.1 apply to the canonical resolution of \( IT'(X) \) by 4.2. A summand of the canonical resolution of \( IT'(X) = IT'(sl(A)) \) has a conical contraction \( c'(A) \) depending functorially on \( A \) by 4.1. After applying the cotensor product \( R \rightleftharpoons T'(X) \) to the above two parts of the canonical resolution for \( R \) over \( T'(X) \) we have the complexes

\[ \ldots 0 \to 0 \to R \xrightarrow{0} X \xrightarrow{0} \ldots, \]

\[ \ldots 0 \to 0 \to 0 \to IT'(X) \xrightarrow{c(A)} IT'(X)^2 \otimes \to \ldots. \]

The second complex desuspended and assembled as the tensor algebra is just

\[ IS\Omega B(A) = IT(s^{-1} T'(X)) = IT(s^{-1} T'(sl(A))). \]

The first term splits canonically to give a projection \( IT'(X) \to X = sl(A) \) or \( s^{-1} IT'(X) \to I(A) \), and this is just \( \alpha(A) \) restricted to \( s^{-1} IT'(X) \to I(A) \). If \( Y \) is the kernel of this map, then \( N(A) \) is the assembled subalgebra of the tensor algebra \( T(s^{-1} T'(X)) \), where \( IN(A) \) is the result of assembling

\[ 0 \to Y \xrightarrow{c(A)} IT'(X)^{2 \otimes} \leftarrow IT'(X)^{3 \otimes} \leftarrow \ldots. \]

Since the sequence

\[ 0 \to X \to IT'(X) \xrightarrow{c(A)} IT'(X)^{2 \otimes} \leftarrow \ldots \]

is exact with canonical splittings, we have the following result.
4.3. Summary. For algebras $A$ with $I(A)^2 = 0$ and $d(A) = 0$ there is a conical contraction $\alpha(A)$ on $IN(A)$ depending functorially on $A$. Moreover, $\alpha(A) : \Omega B(A) \rightarrow A$ is a homotopy equivalence.

Now we wish to reduce the general case to the special case where $I(A)^2 = 0$ and $d(A) = 0$ by a suitable filtration on the algebra $A$, namely $F_p A$, where $F_0 A = R$ and for $i > 0$, the $i-1$ skeleton of $A$. Then the bigraded algebra $E^0(A)$ has the property that $IE^0(A)^2 = 0$ and $dE^0(A) = d^0 = 0$. Moreover, the filtration on $A$ induces a filtration on $B(A)$ such that

$$E^0(B(A)) = B(E^0(A)) = T'(sl(E^0(A))),$$

on $\Omega B(A)$ such that

$$E^0 \Omega B(A) = \Omega E^0 B(A) = \Omega T'(sl(E^0(A))),$$

and on $N(A)$ such that

$$IN(E^0(A)) = IE^0 N(A) = E^0 IN(A)$$

has a conical contraction depending functorially on $A$.

Observe that $IN(E^0(IN(A)))$ is isomorphic to a cocone $C'(Y(p))$, where the isomorphism depends functorially on $A$, and there is a sequence of morphisms

$$h_p : E^0_p(IN(A)) \rightarrow F_p(IN(A))$$

of graded modules such that the composite

$$E^0_p(IN(A)) \rightarrow F_p(IN(A)) \rightarrow E^0_p(IN(A))$$

is the identity. Thus the morphism $Y(p) \rightarrow E^0_p(IN(A))$, which is the restriction of the isomorphism $C'(Y(p)) \rightarrow E^0_p(IN(A))$, composed with $h_p : E^0_p(IN(A)) \rightarrow F_p(IN(A))$ defines by the adjointness properties of $C'$ a morphism $j_p : C'(Y(p)) \rightarrow F_p(IN(A))$ such that the composite

$$C'(Y(p)) \rightarrow F_p(IN(A)) \rightarrow F^0_p(IN(A)) \rightarrow C'(Y(p))$$

is the identity. Thus there is a sequence of isomorphisms

$$f^0_p : F_p(IN(A)) \rightarrow \bigsqcup_i E^0_i(IN(A))$$

depending functorially on $A$ for each $p$ such that $f^0_{p+1}$ is an extension of $f^0_p$. This discussion yields immediately the following theorem, which is an extension of 4.3, using the above isomorphism

$$f : IN(A) \rightarrow \bigsqcup_i E^0_i(IN(A)),$$

where $f \mid F_p(IN(A)) = f^0_p$. 

4.4. Theorem. There is a conical contraction $c(A)$ on $IN(A)$ depending functorially on $A$. Moreover, $\alpha(A) : \Omega B(A) \to A$ is a homotopy equivalence.

If $B\Omega(C) \to N(C)$ is the coalgebra cokernel of $\beta(C) : C \to B\Omega(C)$, then we have the following theorem by the dual argument, where the details are left to the reader.

4.5. Theorem. There is a conical contraction $c(C)$ on $IN(C)$ depending functorially on $C$. Moreover, $\beta(C) : C \to B\Omega(C)$ is a homotopy equivalence.

5. Acyclicity of the universal bundles and extended loop classifying adjointness

We continue to use the notation $\tau(C) : C \to \Omega(C)$ and $\tau(A) : B(A) \to A$ for the canonical twisting morphism, where $C$ is a coalgebra and $A$ is an algebra.

5.1. Definition. (1) For a coalgebra $C$, the left universal bundle of $C$ is the $(C, \Omega(C))$-bundle 

$$E_L(C) = C \otimes_{\tau(C)} \Omega(C),$$

and the right universal bundle of $C$ is the $(\Omega(C), C)$-bundle 

$$E_R(C) = \Omega(C) \otimes_{\tau(C)} C.$$

(2) For an algebra $A$, the right universal bundle of $A$ is the $(B(A), A)$-bundle 

$$W_R(A) = B(A) \otimes_{\tau(A)} A$$

and the left universal bundle of $A$ is the $(A, B(A))$-bundle 

$$W_L(A) = A \otimes_{\tau(A)} B(A).$$

Note that 

$$E_R(R) = R = E_L(R),$$

and that there are natural morphisms 

$$R \to E_L(C), \quad E_L(C) \to R$$

such that the composite is the identity. Similarly, 

$$W_R(R) = R = W_L(R)$$

and the pairs 

$$R \to E_R(C) \to R, \quad R \to W_R(A) \to R, \quad R \to W_L(A) \to R$$

compose to the identity.
5.2. Proposition. (1) For a coalgebra $C$ the morphisms $R \to E^1(C)$ and $R \to E^2(R)(C)$ viewed in $\mathcal{C}_s(R)$ are morphisms split over $R$ which are homotopy equivalences.

(2) For an algebra $A$ the morphisms $W_R(A) \to R$ and $W_L(A) \to R$ viewed in $\mathcal{C}_s(R)$ are epimorphisms split over $R$ which are homotopy equivalences.

Proof. Let

$$T : \mathcal{C}_s(C^\mathfrak{M}_C) \times \mathcal{C}_s(C^\mathfrak{M}_C) \to \mathcal{C}_s(C^R)$$

be the cotensor product assembly functor as in 1, Section 4. As differential $R$-modules, $E^1(C)$ is isomorphic to $F_\infty Ao(X)$, where $X = T(C, U_\mathfrak{M}(R))$. Since the sequence

$$0 \to R \to U_0(R) \to U_1(R) \to \ldots$$

is split exact over $R$, $E^0(X)$ is contractible to $R$. By 4.4 the morphism $R \to E^1(C)$ is a homotopy equivalence in $\mathcal{C}_s(R)$. The same statement holds for $R \to E^2(R)$.

Part (2) of the proposition follows by a dual proof. □

5.3. Notation. Let $\tau : C \to A$ be a twisting morphism for an algebra $A$ and a coalgebra $C$. Let $M$ be a right $A$-module and $Y$ a left $C$-comodule. Let

$$d_\tau : M \otimes Y \to M \otimes Y$$

denote the composite

$$M \otimes Y \xrightarrow{M \otimes \Delta(Y)} M \otimes C \otimes Y \xrightarrow{M \otimes \tau \otimes Y} M \otimes A \otimes Y \xrightarrow{\phi(M) \otimes Y} M \otimes Y.$$

Then $Dd_\tau = -d_\tau^2$ by 1.2 and $M \otimes_\tau Y$ denotes $M \otimes Y$ with the differential $d(M \otimes Y) + d_\tau$.

Let $N$ be a left $A$-module and $X$ a right $C$-comodule. Let

$$d_\tau : X \otimes N \to X \otimes N$$

denote the composite

$$X \otimes N \xrightarrow{\Delta(X) \otimes N} X \otimes C \otimes N \xrightarrow{X \otimes \tau \otimes N} X \otimes A \otimes N \xrightarrow{X \otimes \phi(N)} X \otimes N.$$

Then $Dd_\tau = d_\tau^2$ by 1.2 and $X \otimes_\tau N$ denotes $X \otimes N$ with the differential $d(X \otimes N) - d_\tau$.

5.4. Remarks. For twisting morphisms $\tau : C \to A$ and $\tau' : C \to A'$, a two-sided $C$-comodule $X$, a right $A$-module $M$, and a left $A'$-module $N$, it follows that:

(1) $X \otimes_\tau N$ is a left $C$-comodule,

(2) $M \otimes_\tau X$ is a right $C$-comodule,

(3) $M \otimes_\tau (X \otimes_\tau' N) = (M \otimes_\tau X) \otimes_\tau' N$.

For twisting morphisms $\tau : C \to A$ and $\tau' : C \to A'$, a two-sided $A$-module $M$, a right $C$-comodule $X$, and a left $C'$-comodule $Y$, it follows that:

(1) $X \otimes_\tau M$ is a right $A$-module,

(2) $M \otimes_\tau' Y$ is a left $A$-module,

(3) $(X \otimes_\tau M) \otimes_\tau' Y = X \otimes_\tau (M \otimes_\tau' Y)$. 

D. Husemoller et al., Differential homological algebra
5.5. **Remarks.** For an algebra $A$ we have canonical isomorphisms

$$W^R_R(A) = E^L_L(B(A)) \otimes_{\Omega(B(A))} A, \quad W^L_L(A) = A \otimes_{\Omega(B(A))} E^R_R(B(A)).$$

For a coalgebra $C$ we have canonical isomorphisms

$$E^L_L(C) = C \otimes_{B(\Omega(C))} W^R_R(\Omega(C)), \quad E^R_R(C) = W^L_L(\Omega(C)) \otimes_{B(\Omega(C))} C.$$

5.6. **Notation.** For an algebra $A$, let

$$\tilde{F}(A) = A \otimes_{\tau(A)} B(A) \otimes_{\tau(A)} A.$$  

For a coalgebra $C$, let

$$\tilde{E}(C) = C \otimes_{\tau(C)} \Omega(C) \otimes_{\tau(C)} C.$$

5.7. **Definition.** An extended coalgebra is a triple $(X, C, Y)$ such that $C$ is a coalgebra, $X$ is a right $C$-comodule, and $Y$ is a left $C$-comodule. A morphism $(X', C', Y) \to (X, C, Y)$ of extended coalgebras is a triple $(u, v, w)$, where $u : C' \to C$ is a morphism of coalgebras and $u : X' \to X$ and $w : Y' \to Y$ are morphisms of differential $R$-modules such that

\[
\begin{array}{ccc}
X' & \xrightarrow{\Delta(X')} & X' \otimes C' \\
\downarrow{u} & & \downarrow{u \otimes v} \\
X & \xrightarrow{\Delta(X)} & X \otimes C \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
Y' & \xrightarrow{\Delta(Y')} & C' \otimes Y' \\
\downarrow{w} & & \downarrow{u \otimes w} \\
Y & \xrightarrow{\Delta(Y)} & C \otimes Y \\
\end{array}
\]

are commutative.

The category $\text{Coalg}(R)$ is the category of extended coalgebras and morphisms with composition being induced by composition in $\mathcal{C}_+(R)$.

5.8. **Definition.** An extended algebra is a triple $(L, A, M)$ such that $A$ is an algebra, $L$ is a right $A$-module, and $M$ is a left $A$-module. A morphism $(L, A, M) \to (L'', A'', M'')$ of extended algebras is a triple $(u, v, w)$, where $u : A \to A''$ is a morphism of algebras and $u : L \to L''$ and $w : M \to M''$ are morphisms of differential $R$-modules such that

\[
\begin{array}{ccc}
L \otimes A & \xrightarrow{\phi(L)} & L \\
\downarrow{u \otimes v} & & \downarrow{u} \\
L'' \otimes A'' & \xrightarrow{\phi(L'')} & L'' \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A \otimes M & \xrightarrow{\phi(M)} & M \\
\downarrow{v \otimes w} & & \downarrow{w} \\
A'' \otimes M'' & \xrightarrow{\phi(M'')} & M'' \\
\end{array}
\]

are commutative.

The category $\text{Alg}(R)$ is the category of extended algebras and morphisms with composition being induced by composition in $\mathcal{C}_+(R)$.
The function assigning \((R, C, R)\) to a coalgebra \(C\) is a full embedding \(\text{Coalg}(R) \rightarrow \text{Coalg}(R)\), and the function assigning \((R, A, R)\) to an algebra \(A\) is a full embedding \(\text{Alg}(R) \rightarrow \text{Alg}(R)\).

5.9. Definition. For an extended coalgebra \((X, C, Y)\) define the extended loop algebra \(\Omega(X, C, Y)\), also denoted \((\Omega^C(X), \Omega(C), \Omega^C(Y))\), to be
\[
(X \ominus_C E_L(C), \Omega(C), E_R(C) \ominus_C Y).
\]
For a morphism \((u, v, w) : (X', C', Y') \rightarrow (X, C, Y)\) of extended coalgebras define
\[
\Omega(u, v, w) = (u \ominus_C E_L(v), \Omega(v), E_R(v) \ominus_C w).
\]
where a slight abuse of notation is used.

5.10. Definition. For an extended algebra \((L, A, M)\) define the extended classifying coalgebra \(B(L, A, M)\), also denoted \((B_A(L), B(A), B_A(M))\), to be
\[
(L \otimes_A W_L(A), B(A), W_R(A) \otimes_A M).
\]
For a morphism \((u, v, w) : (L, A, M) \rightarrow (L', A', M')\) of extended algebras define
\[
B(u, v, w) = (u \otimes_A W_L(v), B(v), W_R(v) \otimes_A w),
\]
where again a slight abuse of notation is used.

For an algebra \(A\) we denote by \(\theta(A) : \tilde{\Omega}(A) \rightarrow A\) the composite
\[
\phi(A)(A \otimes \varepsilon(B(A)) \otimes A):
\]
\[
\tilde{\Omega}(A) = A \otimes_{\varepsilon(A)} B(A) \otimes_{\varepsilon(A)} A \rightarrow A \otimes R \otimes A = A \otimes A \rightarrow A.
\]
For a coalgebra \(C\) we also denote by \(\theta(C) : C \rightarrow \tilde{\Omega}(C)\) the composite
\[
(C \otimes \varepsilon(\Omega(C)) \otimes C) \varepsilon(C):
\]
\[
C \rightarrow C \otimes C = C \otimes R \otimes C \rightarrow C \otimes \varepsilon(C) \otimes (\Omega(C) \otimes \varepsilon(C)) C = \tilde{\Omega}(C).
\]

5.11. Proposition. (1) For an algebra \(A\) the morphism \(\theta(A) : \tilde{\Omega}(A) \rightarrow A\) is a morphism of two-sided \(A\)-modules which is a homotopy equivalence of either left or right \(A\)-modules.

(2) For a coalgebra \(C\) the morphism \(\theta(C) : C \rightarrow \tilde{\Omega}(C)\) is a morphism of two-sided \(C\)-comodules which is a homotopy equivalence of either left or right \(C\)-comodules.

Proof. For (1), check first that \(\theta(A)\) is a morphism of two-sided \(A\)-modules. Then filter \(\tilde{\Omega}(A)\) by letting \(F_p(\tilde{\Omega}(A)) = F_p(W_L(A)) \otimes A\), where \(F_p W_L(A)\) is the \(p\)-skeleton of \(W_L(A)\), and filter \(A\) by skeletons. With these filtrations,
\[
E^0(\tilde{\Omega}(A)) = W_L(A) \otimes A.
\]
\( \theta(A) \) is filtration preserving, and
\[
E^0(\theta(A)) : E^0(\tilde{W}(A)) \to E^0(A)
\]
is a homotopy equivalence in \( \mathcal{C}_+(R) \) by 5.2. It follows readily that \( \theta(A) \) is a homotopy equivalence of right \( A \)-modules. The fact that it is a homotopy equivalence of left \( A \)-modules follows similarly. Part (2) follows by a dual argument. □

5.12. Remark. For an extended algebra \((L, A, M)\), one has
\[
(\Omega B(A)(B_A(L)) \otimes_{\Omega B(A)} A, A \otimes_{\Omega B(A)} \Omega B(A)(B_A(M))) = (L \otimes_A \tilde{W}(A), A, \tilde{W}(A) \otimes_A M).
\]
For an extended coalgebra \((X, C, Y)\), one has
\[
(\Omega \epsilon(C)(\Omega \epsilon(X)) \circ_{\Omega \epsilon(C)} C, C \circ_{\Omega \epsilon(C)} \Omega \epsilon(C) (\Omega \epsilon(Y))) = (X \circ_{C} E(C), C, E(C) \circ_{C} Y).
\]

5.13. Definition. For an extended algebra \((L, A, M)\) let
\[
\alpha(L, A, M) : \Omega B(L, A, M) \to (L, A, M)
\]
be the morphism \((\alpha_A(L), \alpha(A), \alpha_A(M))\), where \(\alpha(L) : \Omega B(A) \to A\) is the adjunction morphism for \( \Omega \dashv B \), \(\alpha_A(L)\) is the composite
\[
\Omega B(A)(B_A(L)) \to \Omega B(A)(L) \otimes_{\Omega B(A)} A = L \otimes_A \tilde{W}(A) \xrightarrow{L \otimes_A \theta(A)} L \otimes_A A = L,
\]
and \(\alpha_A(M)\) is the composite
\[
\Omega B(A)(B_A(M)) \to A \otimes_{\Omega B(A)} \Omega B(A)(M) = \tilde{W}(A) \otimes_A M \xrightarrow{\theta(A) \otimes_A M} A \otimes_A M = M.
\]

5.14. Definition. For an extended coalgebra \((X, C, Y)\) let
\[
\beta(X, C, Y) : (X, C, Y) \to B\Omega(X, C, Y)
\]
be the morphism \((\beta_C(X), \beta(C), \beta_C(Y))\), where \(\beta(C) : C \to B\Omega(C)\) is the adjunction morphism for \( \Omega \dashv B \), \(\beta_C(X)\) is the composite
\[
X = X \circ_{\Omega} C \xrightarrow{X \circ_{\Omega} \theta(C)} X \circ_{\Omega} E(C) = B_{\Omega(C)}(\Omega_C(X)) \circ_{B\Omega(C)} C \to B_{\Omega(C)}(\Omega_C(X)).
\]
and \(\beta_C(Y)\) is the composite
\[
Y = C \circ_{\Omega} Y \xrightarrow{\alpha(C) \circ Y} E(C) \circ_{\Omega} Y = C \circ_{B\Omega(C)} B_{\Omega(C)}(\Omega_C(Y)) \to B_{\Omega(C)}(\Omega_C(Y)).
\]

5.15. Theorem. The above morphisms and functors yield an adjoint pair
\[
(\alpha, \beta) : \Omega \dashv B : \overline{\text{Alg}}(R) \to \overline{\text{Coalg}}(R).
\]
Moreover,

(1) for an extended algebra \((L, A, M)\) the morphism \(\alpha(L, A, M)\) is a triple of morphisms in \( \mathcal{C}_+(R) \) each of which is an \( R \)-split epimorphism and a homotopy equivalence,
(2) for an extended coalgebra \((X, C, Y)\) the morphism \(\beta(X, C, Y)\) is a triple of morphisms in \(\mathfrak{C}_*(R)\) each of which is an \(R\)-split monomorphism and a homotopy equivalence.

**Proof.** One checks adjointness easily. Then

\[
\Omega_{B(A)}(B_A(L)) = L \otimes_{\tau(A)} B(A) \otimes_{\tau(B(A))} \Omega B(A).
\]

Filter this by

\[
F_p \Omega_{B(A)}(B_A(L)) = F_p (L \otimes_{\tau(A)} B(A) \otimes_{\tau(B(A))} \Omega B(A)),
\]

where \(F_p(L \otimes_{\tau(A)} B(A))\) is the \(p\)-skeleton. Then it follows that

\[
E^0(\Omega_{B(A)}(B_A(L))) = E^0(B_A(L) \otimes \Omega B(A)).
\]

Filtering \(L \otimes_A \tilde{W}(A)\) similarly, we have also that

\[
E^0(L \otimes_A \tilde{W}(A)) = E^0(B_A(L)) \otimes A.
\]

Since \(\Omega B(A) \to A\) is a split epimorphism which is a homotopy equivalence, it follows that

\[
E^0(\Omega_{B(A)}(B_A(L))) \to E^0(L \otimes_A \tilde{W}(A))
\]

is a split epimorphism which is a homotopy equivalence, so

\[
\Omega_{B(A)}(B_A(L)) \to L \otimes_A \tilde{W}(A)
\]

is also. Using 5.11, \(L \otimes_A \tilde{W}(A) \to L\) is a split epimorphism which is a homotopy equivalence. This proves that \(\alpha_A(L)\) has this property, and a similar argument shows that \(\alpha_A(M)\) has this property. Thus each morphism in the triple

\[
\alpha(L, A, M) = (\alpha_A(L), \alpha(A), \alpha_A(M))
\]

is a split epimorphism which is a homotopy equivalence.

Part (2) is established by dualizing the argument used for part (1), and the theorem is proved. \(\square\)
III. Simplicial sets and fibrations

This part is devoted to outlining those properties of chains on simplicial sets which are needed to study the homological properties of fibrations. In particular we reconsider both the Serre calculation of \( E^1 \) and \( E^2 \) of the spectral sequence of the filtration coming from a map and the Eilenberg–Zilber theorem. In this way we move to the geometric setting for the homological algebra developed in the previous two parts.

We assume that the reader is familiar with the basic definitions in the theory of simplicial sets, see for example exposés by J. Moore in [9].

1. Simplicial sets, \( \text{Map}(X, Y) \) and homotopies

For each natural number \( n \) let \( \Delta_n \) denote the simplicial set which is the standard \( n \)-simplex, and let \( i_n \) be the standard nondegenerate \( n \)-simplex in \( (\Delta_n)_n \). The function which assigns to each morphism of simplicial sets \( f : \Delta_n \to X \) the element \( f(i_n) \in X_n \) is a bijection onto the set \( X_n \). Frequently we will make this identification between the singular \( n \)-simplexes of \( X \), i.e. morphisms \( \Delta_n \to X \), and elements of \( X_n \).

We denote the category of simplicial sets by \( \Delta(\text{sets}) \) and more generally the category of simplicial objects over a category \( \mathcal{X} \) by \( \Delta(\mathcal{X}) \). The product of simplicial sets \( X \times Y \) has the property that \( (X \times Y)_n = X_n \times Y_n \) as sets.

1.1. Definition. For two simplicial sets \( X \) and \( Y \), the mapping simplicial set \( \text{Map}(X, Y) \) is defined by

\[
\text{Map}(X, Y)_n = \Delta(\text{sets}) (\Delta_n \times X, Y).
\]

For any morphism \( \sigma : \Delta_p \to \Delta_n \) we define

\[
\sigma^* : \text{Map}(X, Y)_n \to \text{Map}(X, Y)_p
\]

by the relation

\[
\sigma^* (f) = f(\sigma \times X).
\]

In particular,

\[
\text{Map}(X, Y)_0 = \Delta(\text{sets}) (X, Y).
\]

Further,

\[
\text{Map} : \Delta(\text{sets})^* \times \Delta(\text{sets}) \to \Delta(\text{sets})
\]

is a functor.

Moreover, \( \text{Map}(X, Y) \) has the property that there is a natural isomorphism

\[
\text{Map}(X \times Y, Z) \to \text{Map}(X, \text{Map}(Y, Z)).
\]

In the language of category theory, \( \Delta(\text{sets}) \) is a Cartesian closed category with internal morphism functor \( \text{Map} \).
A path (edge path) between two 0-simplexes $x, x' \in X_0$ is a sequence $x = x_0, x_1, \ldots, x_m = x'$ in $X_0$ and a sequence $c_1, \ldots, c_m \in X_1$ of 1-simplexes such that

$$d_0 c_i = x_{i-1}, \quad d_1 c_i = x_i$$
onumber

Two zero simplexes being connected by an edge path is an equivalence relation on $X_0$.

1.2. Definition. Two morphisms $f, g : X \rightarrow Y$ between simplicial sets are homotopic provided there is an edge path in $\text{Map}(X, Y)$ between $f$ and $g$.

For a simplicial set $X$, let $C_\ast(X; M)$ denote the chains of $X$ with coefficients in the $R$-module $M$. Then

$$C_\ast(-; M) : \Delta(\text{sets}) \rightarrow \mathcal{C}_\ast(R)$$

is a functor. A homotopy between $f, g : X \rightarrow Y$ induces a chain homotopy between $C_\ast(f; M), C_\ast(g; M) : C_\ast(X, M) \rightarrow C_\ast(Y, M)$. Hence it follows that

$$H_\ast(f; M) = H_\ast(g; M).$$

1.3. Remark. For a pair of simplicial sets $(X, A)$ there is a cocartesian square

$$\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & X/A
\end{array}$$

where $\ast$ is the simplicial point. The functor $(X, A) \leftarrow X/A$ is the functor coadjoint to the natural inclusion functor from the category of pointed simplicial sets to the category of pairs of simplicial sets. A morphism $f : (X, A) \rightarrow (Y, B)$ of pairs of simplicial sets is an excision morphism if $\tilde{f} : X/A \rightarrow Y/B$ is an isomorphism of pointed simplicial sets. In this case

$$C_\ast(f; M) : C_\ast(X, A; M) \rightarrow C_\ast(Y, B; M)$$

and hence

$$H_\ast(f, M) : H_\ast(X, A; M) \rightarrow H_\ast(Y, B; M)$$

are isomorphisms.

1.4. Definition. The $p$-skeleton $F_p X$ of a simplicial set $X$ is empty for $p < 0$ and $(F_p X)_n$ consists of all $\Delta_n \rightarrow X$ which factor $\Delta_n \rightarrow \Delta_p \rightarrow X$. 
Clearly \((F_p X)_n = X_n\) for \(p > n\), and we filter the chains by
\[
F_p C_\ast(X; M) = C_\ast(F_p X; M).
\]
This is the filtration by skeletons.

1.5. Remarks. Then \(E^{1}_{p,0} C_\ast(X)\) is a quotient of \(C_p(X)\), denoted \(C_p^{\infty}(X)\), called the normalized chain complex, and
\[
E^{1}_{p,q} C_\ast(X) = 0 \quad \text{for} \quad q \neq 0.
\]
A simplex \(x \in X_n\) is nondegenerate provided \(x \notin F_{n-1}(X)_n\). The nondegenerate simplexes project to a basis of \(C_n^{\infty}(X)\). Since \(E^{1}_{p,q} C_\ast(X) = 0 \quad \text{for} \quad q \neq 0\), we have
\[
E^{2}_{p,0} C_\ast(X) = H_p(X).
\]
and the spectral sequence collapses.

2. Coalgebra and comodule structures on chain modules

For a simplicial set \(X\), the set \(X_n\) is a basis for \(C_n(X)\) (the unnormalized chains of \(X\)). Let
\[
\Delta(C_\ast(X)) : C_\ast(X) \to C_\ast(X) \otimes C_\ast(X)
\]
be the Alexander–Čech–Whitney morphism. For \(\Delta(C_\ast(X))\) abbreviated to \(\Delta\), we have for \(v \in X_n\),
\[
\Delta(v) = \sum_{0 < j < n} \tilde{d}^j v \otimes d_0^{n-j} v,
\]
where \(\tilde{d}\) is the last face operator and \(d_0\) is the 0th face operator. Let \(\epsilon : C_\ast(X) \to R\) be the morphism such that \(\epsilon(v) = 1\) for \(v \in X_0\). The triple \((C_\ast(X), \Delta, \epsilon)\) is a coalgebra.

2.1. Remark. The triple \((C_\ast(X), \Delta, \epsilon)\) is a coalgebra, and \(C_\ast : \Delta(\text{sets}) \to \text{Coalg}'(R)\) is a functor, where \(\text{Coalg}'(R)\) is the category of all coalgebras (not just connected ones). Moreover, when \(C_\ast(X)\) is filtered by \(F_p C_\ast(X) = C_\ast(F_p X)\), where \(F_p X\) is the \(p\)-skeleton of \(X\), the morphism \(\Delta(C_\ast(X))\) is filtration preserving.

For the latter statement recall that an \(n\)-simplex of \(\Delta_p\) is an increasing sequence \((i_0, ..., i_n)\) of integers between 0 and \(p\) such that
\[
\tilde{d}^j(i_0, ..., i_n) = (i_0, ..., i_{n-j}).
\]
\[
(d_0)^{n-j} (i_0, ..., i_n) = (i_{n-j}, ..., i_n).
\]
Now the numbers of distinct vertices in \((i_0, ..., i_n)\) and in \((i_0, ..., i_{n-j})\) together with \((i_{n-j}, ..., i_n)\) are the same.
Let $X$ and $Y$ be simplicial sets. Let $X \otimes Y$ denote the filtered simplicial set such that

$$F_\infty(X \otimes Y) = X \times Y$$

and

$$F_p(X \otimes Y) = \bigcup_{r+s=p} F_r(X) \times F_s(Y),$$

where $X$ and $Y$ are filtered by skeletons. Filtering $X \times Y$ by skeletons, we have a unique morphism $i(\mathcal{V}) : X \otimes Y \to X \times Y$ such that $F_\infty(i(\mathcal{V}))$ is the identity on $F_\infty(X \otimes Y) = X \times Y$.

Next $C_\bullet(X \otimes Y)$ is a filtered coalgebra such that

$$F_p C_\bullet(X \otimes Y) = C_\bullet(F_p(X \otimes Y)).$$

$C_\bullet(i(\mathcal{V}))$ is a morphism of filtered coalgebras, and

$$E^0 C_\bullet(i(\mathcal{V})) : E^0(C_\bullet(X \otimes Y)) \to E^0(C_\bullet(X \times Y))$$

is a morphism of coalgebras. Looking at basis elements, one sees easily that

$$E^0 C_\bullet(X \otimes Y) = E^0 C_\bullet(X) \otimes E^0 C_\bullet(Y),$$

and this implies that

$$E^1 C_\bullet(X \otimes Y) = C^N_\bullet(X) \otimes C^N_\bullet(Y).$$

Thus one has that

$$E^1 C_\bullet(i(\mathcal{V})) : C^N_\bullet(X) \otimes C^N_\bullet(Y) \to C^N_\bullet(X \times Y)$$

is a morphism of coalgebras.

**2.2. Remark.** The morphism $E^1 C_\bullet(i(\mathcal{V}))$ is

$$\mathcal{V} : C^N_\bullet(X) \otimes C^N_\bullet(Y) \to C^N_\bullet(X \times Y),$$

the subdivision (or shuffle) morphism of the Eilenberg–Zilber theorem; see [12].

The other morphism $f$ of the Eilenberg–Zilber theorem is the composite

$$C^N_\bullet(X \times Y) \xrightarrow{\Delta C^N_\bullet(X \times Y)} C^N_\bullet(X \times Y) \otimes C^N_\bullet(X \times Y) \xrightarrow{C^N_\bullet(pr_X) \otimes C^N_\bullet(pr_Y)} C^N_\bullet(X) \otimes C^N_\bullet(Y),$$

which has the property that $f \mathcal{V} = f E^1 C_\bullet(i(\mathcal{V}))$ is the identity on $C^N_\bullet(X) \otimes C^N_\bullet(Y)$.

The preceding discussion leads to an alternative proof of the Eilenberg–Zilber theorem together with the addendum that the subdivision morphism of coalgebras; see [13, §18]. Note that it is rarely the case that $f$ is a morphism of coalgebras. Essentially this only occurs if one of $X$ and $Y$ is discrete, in which case $f$ is an isomorphism.
3. The category $\text{Mor}(\Delta(\text{sets}))$

For any category $\mathcal{X}$ let $\text{Mor}(\mathcal{X})$ denote the category whose objects are morphism $f : X' \to X$ in $\mathcal{X}$ and whose morphisms $(u, v) : f \to g$ are pairs of morphisms $(u, v)$ in $\mathcal{X}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow f & & \downarrow g \\
X & \longrightarrow & Y
\end{array}
$$

Composition in $\text{Mor}(\mathcal{X})$ is induced by composition of pairs in $\mathcal{X}$. Note that $\mathcal{X}$ can be identified with the full subcategory of $\text{Mor}(\mathcal{X})$ determined by the identity morphisms in $\mathcal{X}$.

Clearly, for $f : X' \to X$ and $g : Y' \to Y$ in $\text{Mor}(\mathcal{X})$,

$$
(f \circ g) : X' \circ Y' \to X \circ Y
$$

is the product of $f$ and $g$ in the category $\text{Mor}(\mathcal{X})$.

3.1. Definition. For $f : X' \to X$ and $g : Y' \to Y$ in $\text{Mor}(\Delta(\text{sets}))$, the mapping simplicial set $\text{Map}(f, g)$ is defined by

$$
\text{Map}(f, g)_n = \text{Mor}(\Delta(\text{sets}))(\Delta_n \times f, g).
$$

For any morphism $\sigma : \Delta_p \to \Delta_n$ we define $\sigma^* : \text{Map}(f, g)_n \to \text{Map}(f, g)_p$ by the relation

$$
\sigma^*(u, v) = (u(\sigma \times X'), v(\sigma \times X)).
$$

In particular,

$$
\text{Map}(f, g)_0 = \text{Mor}(\Delta(\text{sets}))(f, g).
$$

Further,

$$
\text{Map} : \text{Mor}(\Delta(\text{sets}))^\bullet \times \text{Mor}(\Delta(\text{sets})) \to \Delta(\text{sets})
$$

is a functor. In the language of category theory, $\text{Mor}(\Delta(\text{sets}))$ is enriched over the cartesian closed category $\Delta(\text{sets})$.

3.2. Definition. Two morphisms $(u, v), (u', v') : f \to g$ between morphisms of simplicial sets are homotopic provided there is an edge path in $\text{Map}(f, g)$ between $(u, v)$ and $(u', v')$.

3.3. Definition. Let $\pi : E \to B$ be a morphism of simplicial sets, i.e. and object of $\text{Mor}(\Delta(\text{sets}))$. The $n$-inverse image filtration $F^\pi_n E$ on $E$ is defined by $\pi^{-1}(F^\pi_p B)$, where $F^\pi_p B$ is the skeleton filtration on $B$. 
In other words, the following diagram is Cartesian:

\[
\begin{array}{ccc}
F_p^\pi E & \to & E \\
\downarrow & & \downarrow \\
F_p B & \to & B
\end{array}
\]

For an \( R \)-module \( M \) we define

\[
C_* (\cdot : M) : \text{Mor}(\Delta(\text{sets})) \to \mathcal{C} \mathcal{T} \cup \mathcal{C}_+(R),
\]

where for \( \pi \) in \( \text{Mor}(\Delta(\text{sets})) \) the filtered differential \( R \)-module is defined by

\[
F_\infty C_* (\pi : M) = C_* (E, M), \quad F_p C_* (\pi : M) = C_* (F^\pi_p E : M).
\]

For a morphism \((u, v) : \pi \to \pi'\) we have

\[
\pi(F^\pi_p E) \subset F^\pi_p E',
\]

since \( \pi(F^\pi_p E) \subset F^\pi_p B' \), and thus \( C_* (u, v) : C_* (\pi, M) \to C_* (\pi', M) \) is defined. The \( \pi \)-inverse image filtration is regular.

3.4. Remark. If two morphisms \((u, v), (u', v') : \pi \to \pi'\) are homotopic, then \( C_* (u, v) \) and \( C_* (u', v') : C_* (\pi, M) \to C_* (\pi', M) \) are homotopic in \( \mathcal{C} \mathcal{T} \cup \mathcal{C}_+(R) \); see 1, 3.5.

Before calculating \( E^1(\pi : M) \) in the next section, we consider briefly the Eilenberg–Zilber theorem in the context of \( \text{Mor}(\Delta(\text{sets})) \).

Let \( \pi_1 : E_1 \to B_1 \) and \( \pi_2 : E_2 \to B_2 \) be two morphisms of simplicial sets, and form the product \( \pi_1 \times \pi_2 = \pi : E \to B \), where \( E = E_1 \times E_2 \) and \( B = B_1 \times B_2 \). The tensor product

\[
\pi_1 \otimes \pi_2 : E_1 \otimes E_2 \to B_1 \otimes B_2
\]

is the morphism of filtered simplicial sets, where

\[
F_\infty (B_1 \otimes B_2) = B_1 \times B_2, \quad F^\otimes_p (B_1 \otimes B_2) = \coprod_{r+s=p} F_r (B_1) \times F_s (B_2)
\]

\[
F_\infty (E_1 \otimes E_2) = E_1 \times E_2, \quad F^\otimes_p (E_1 \otimes E_2) = \pi^{-1}(F^\otimes_p (B_1 \otimes B_2))
\]

In other words, the following diagram is Cartesian:

\[
\begin{array}{ccc}
F^\otimes_p (E_1 \otimes E_2) & \to & F^\otimes_\infty (E_1 \otimes E_2) = E_1 \times E_2 \\
\downarrow & & \downarrow \\
F^\otimes_p (B_1 \otimes B_2) & \to & F^\otimes_\infty (B_1 \otimes B_2) = B_1 \times B_2
\end{array}
\]
Let $C_*(\pi_1 \otimes \pi_2, M)$ be the filtered differential $R$-module such that
\[ F_p^2 C_*(\pi_1 \otimes \pi_2, M) = C_*(F_p^2(E_1 \otimes E_2), M), \]
and observe that there is a natural subdivision morphism
\[ V : C_*(\pi_1 \otimes \pi_2) \to C_*(\pi_1 \times \pi_2). \]

Moreover, the Alexander–Čech–Whitney morphism of $C_*(E_1 \times E_2)$ induces one for $C_*(\pi_1 \times \pi_2)$, namely
\[ \Delta C_*(\pi_1 \times \pi_2) : C_*(\pi_1 \times \pi_2) \to C_*(\pi_1 \times \pi_2) \otimes C_*(\pi_1 \times \pi_2), \]
where $F_\pi \Delta C_*(\pi_1 \times \pi_2) = \Delta C_*(E_1 \times E_2)$. This composed with $C_*(\pi_1 \times \pi_2) \to C_*(\pi_1)$ induced by the projection $\pi_1 \times \pi_2 \to \pi_1$ tensored with $C_*(\pi_1 \times \pi_2) \to C_*(\pi_2)$ induced by the projection $\pi_1 \times \pi_2 \to \pi_2$ yields the morphism
\[ f : C_*(\pi_1 \times \pi_2) \to C_*(\pi_1) \otimes C_*(\pi_2). \]

4. Calculation of the $E^1$-term of $C_*(\pi; M)$

Recall that for $\pi : E \to B$ a morphism of simplicial sets viewed as an object in Mor($\Delta(\text{sets})$) and a morphism $f : B' \to B$ of simplicial sets, there is a Cartesian square in $\Delta(\text{sets})$:
\[
\begin{array}{ccc}
E_f & \xrightarrow{\tilde{f}} & E \\
\downarrow^{\pi_f} & & \downarrow^\pi \\
B' & \xrightarrow{f} & B
\end{array}
\]

Then $(\tilde{f}, f) : \pi_f \to \pi$ is a morphism in Mor($\Delta(\text{sets})$). For the special case $F_p B' = B'$ the morphism $f$ factors through $F_p B$, $\tilde{f}$ factors through $F_p E$, and $F_p E_f = E_f$.

Let $B_n$ denote the subset of nondegenerate $n$-simplexes in $B_n$ for a simplicial set $B$.

For $u \in B_n$, where $u : \Delta_n \to B$, we denote by $\hat{u} : \hat{\Delta}_n \to B$ the restriction of $u$ to the boundary $\hat{\Delta}_n = F_{n-1} \Delta_n$ of $\Delta_n$. When $u \in B_n$ we consider the Cartesian square in the category of pairs of simplicial sets:
\[
\begin{array}{ccc}
(E_u, E_{\hat{u}}) & \xrightarrow{\pi_u} & (F_p^n E, F_{p-1}^n E) \\
\downarrow & & \downarrow \\
(\Delta_n, \hat{\Delta}_n) & \xrightarrow{\hat{u}} & (F_p B, F_{p-1} B)
\end{array}
\]
This gives rise to the following commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{u \in B_p} (E_u, E_u) & \xrightarrow{\bigoplus_{u \in B_p} u} & (F_p^\infty, F_p^\infty E) \\
\downarrow & & \downarrow \\
\bigoplus_{u \in B_p} (\Delta_p, \Delta_p) & \xrightarrow{\bigoplus_{u \in B_p} u} & (F_p B, F_p B)
\end{array}
\]

4.1. Remark. The above morphisms \( \bigoplus_{u \in B_p} u \) and \( \bigoplus_{u \in B_p} \tilde{u} \) are excision morphisms. From this we have an isomorphism of differential \( R \)-modules

\[
\bigoplus_{u \in B_p} \mathbb{C}^\infty(E_u, E_u; M) \xrightarrow{\varepsilon} \mathbb{E}^0_{p,q}(C_u(\pi; M))
\]

Let \( \Delta_p^j \) denote the boundary of \( \Delta_p \) minus the \( j \)th face for \( 0 \leq j \leq p \). Thus we have a morphism of triples

\[
(E_u, E_u, E_u) \rightarrow (\Delta_p, \Delta_p, \Delta_p^j),
\]

where \( u^j \) is \( u \) restricted to \( \Delta_p^j \). Further, the natural excision morphism

\[
e^j: (\Delta_{p-1}, \Delta_{p-1}) \rightarrow (\Delta_p, \Delta_p^j)
\]

induces an excision morphism

\[
ed^j: (E_{d^j u}, E_{d^j u}; M) \rightarrow (E_{u^j}, E_{u^j})
\]

Let

\[
d_f(u): H_{p+q}(E_u, E_u; M) \rightarrow H_{(p-1)+q}(E_{d^j u}, E_{d^j u}; M)
\]

be the composite

\[
H_{p+q}(E_u, E_u; M) \xrightarrow{\alpha} H_{p+q-1}(E_u, E_u; M) \xrightarrow{(-1)^{j} H(e^j u)^{-1}} H_{(p-1)+q}(E_{d^j u}, E_{d^j u}; M)
\]

4.2. Remark. The preceding calculations show that the homology functor applied to the isomorphism in 4.1 yields an isomorphism

\[
\bigoplus_{u \in B_p} H_{p+q}(E_u, E_u; M) \xrightarrow{i(u)} \mathbb{E}^1_{p,q}(C_u(\pi; M))
\]

and if

\[
i(u): H_{p+q}(E_u, E_u; M) \rightarrow \mathbb{E}^1_{p,q}(C_u(\pi; M))
\]
is the canonical injection, then the differential $d^1$ is determined by the formula

$$d^1 i(u) = \sum_{0 \leq j \leq p} i(d_j u) d_j(u)$$

for $u \in \tilde{B}_p$, where $i(d_j u)$ is to be interpreted as zero if $d_j u$ is degenerate.

Supposing that $0 \leq i \leq j \leq p$ and taking into account the fact that under these conditions $d_j d_j = d_{j-1} d_j$, there is a commutative diagram

$$\begin{array}{ccc}
0 & \to & C_*(E_{d_j u}, E_{(d_j u)^i}; M) \\
\downarrow & & \downarrow \\
0 & \to & C_*(E_{d_j u}, E_{(d_j u)^i}; M) \\
\downarrow & & \downarrow \\
0 & \to & C_*(E_{d_j u}, E_{(d_j u)^i}; M) \\
\downarrow & & \downarrow \\
0 & \to & C_*(E_{d_j u}, E_{(d_j u)^i}; M)
\end{array}$$

in $\mathcal{C}(R)$ with exact rows and columns. This implies that the composite connecting morphism taken from the top row and right-hand column is the negative of that taken from the left-hand column and bottom row. As a consequence,

$$d_j d_j = d_{j-1} d_j : H_{p+q}(E_{d_j u}, E_{d_j u}; M) \to H_{(p-2)+q}(E_{d_j d_j u}, E_{(d_j d_j u)}; M).$$

Thus $E_{p,q}^1 (C_*; M)$ is just the normalized chains of $B$ with coefficients in a generalized local coefficient system on $B$ determined by $\pi$ and $M$. This completes the analysis of the prespectral term of the Serre spectral functor on $\text{Mor}(\Delta(\text{sets}))$.

4.3. Remark. For $\pi$ equal the identity on $B$,

$$E_{p,q}^1 (C_*; M) = \begin{cases} 
C_*^N(B; M) & \text{for } q = 0, \\
0 & \text{for } q \neq 0.
\end{cases}$$

This affords yet another proof that either chains or normalized chains may be used to define homology on $\Delta(\text{sets})$, and that when this category is viewed as a full subcategory of $\text{Mor}(\Delta(\text{sets}))$, the natural chains to use are the normalized ones.

5. The spectral sequence of a fibration

5.1. Definition. A simplicial map $\pi : E \to B$ is an $R$-fibration if it is surjective and $H_*(E_u, E_{d_j u}; R) = 0$ for each $u \in B_p$ and $0 \leq i \leq p$. (We are using the notation $E_u$, etc., introduced in the previous section.)
The condition implies that $H_*(E_u, E_d; M) = 0$ for any $R$-module $M$ and a little calculation shows that it also implies that $H_*(E_u, E_u; M) = 0$. Hence

$$d_j(u) = H_{p+q}(E_u, E_u; M) \rightarrow H_{(p-1)+q}(E_d, E_d; M)$$

is an isomorphism for $p > 0$, and letting $u_j$ denote the composite of $u$ with the $j^{th}$ vertex of $\Delta_p$,

$$H_{p+q}(E_u, E_u; M) \rightarrow H_{q}(E_u; M) \rightarrow H_{q}(E_u; M),$$

and this isomorphism is independent of $j$.

5.2. Remark. We have an isomorphism of graded $R$-modules

$$\bigcup_{u \in R_p} H_q(E_u; M) \rightarrow E_{p,q}(C_*(\pi; M)).$$

and if $B$ is connected, $E_{p,q}(C_*(\pi; M))$ is the normalized chains of $B$ with coefficients in a classical local coefficient system.

If $B$ is not connected, then $\pi$ splits as a coproduct in $\text{Mor}(\Delta(\text{sets}))$, each factor of which has for base a connected component of $B$. The functor $C_*(\: ; \: M)$ on $\text{Mor}(\Delta(\text{sets}))$ preserves coproducts, and the object $\pi$ is an $R$-fibration if and only if each component is an $R$-fibration.

Let $\pi : E \rightarrow B$ be an $R$-fibration. If $f$ is a path in $B$ from $b'$ to $b''$, then $f$ determines an isomorphism

$$f : H_*(E_{b'}; M) \rightarrow H_*(E_{b''}; M).$$

5.3. Definition. The $R$-fibration $\pi$ is simple at $M$ if whenever $f$ and $g$ are two paths in $B$ from $b'$ to $b''$, then $f = g : H_*(E_{b'}; M) \rightarrow H_*(E_{b''}; M)$. It is simple if it is simple at $M$ for every $R$-module $M$.

It is always the case that $\pi$ is simple when $B$ is simply connected. This yields a new derivation of the following theorem.

5.4. Theorem (Serre). Let $\pi : E \rightarrow B$ be an $R$-fibration which is simple at $M$ and suppose $B$ is connected with $b \in B$. Then

$$E_{*q}^1(C_*(\pi; M)) = C_{\*}^N(B) \otimes H_*(E_b; M)$$

in $\mathcal{C}_*(R)$, where $C_{\*}^N(B)$ denotes the normalized chains of $B$ with coefficients in $R$. The natural morphism $\pi \rightarrow B \times \pi$ in $\text{Mor}(\Delta(\text{sets}))$ induces

$$E^1C_*(\pi; M) \rightarrow C_{\*}^N(B) \otimes E^1C_*(\pi; M),$$

and $E_{*q}^1(C_*(\pi; M))$ is an extended left $C_{\*}^N(B)$-comodule. Hence

$$E^1(C_*(\pi; M)) = C_{\*}^N(B) \otimes H_*(E_b; M)$$

as $C_{\*}^N(B)$-comodules, and this identification is independent of $b$ up to natural equivalence.
5.5. Remark. Let \( \pi_1 \) and \( \pi_2 \) be simple \( R \)-fibrations with connected bases, and let \( b_1 \in B_1 \) and \( b_2 \in B_2 \). Then

\[
E^1 C_*(\pi_1 \otimes \pi_2 ; M) = C^N_*(B_1) \otimes C^N_*(B_2) \oplus H_*(E_{b_1} \times E_{b_2} ; M).
\]

\( E^1 (V) \) is the tensor product of the subdivision morphism of the Eilenberg–Zilber theorem with \( H_*(E_{b_1} \times E_{b_2} ; M) \), and \( E^1 (f) \) may be identified with the other morphism of the Eilenberg–Zilber theorem.

5.6. Theorem (Eilenberg–Moore). Let \( \pi : E \rightarrow B \) be an \( R \)-fibration which is simple at \( M \), and suppose \( B \) is connected with \( b \in B \), and let \( f : B' \rightarrow B \) be a map with \( \pi_f : E_f \rightarrow B' \) the induced fibration. Then the natural map

\[
H_*(E_f ; M) \rightarrow \text{Cotor} C_*(B) (C_*(B'), C_*(E ; M))
\]

is an isomorphism.

**Proof.** This follows immediately from 5.4 and 1, 6.6. \( \square \)

Finally, there is an isomorphism

\[
C^N_*(B) \otimes H_*(E_b ; M) \rightarrow H_*(E_b ; M) \otimes C^N_*(B),
\]

given by

\[
x_p \otimes y q \mapsto (-1)^{pq} y q \otimes x_p.
\]

If one desires to describe \( E^1 (C_*(\pi ; M)) \) directly in the form \( H_*(E_b ; M) \otimes C^N_*(B) \), then

\[
d^1(lq)(u) : H_{p+q}(E_{u'}, E_{u} ; M) \rightarrow H_{(p+1)+q}(E_{dju'}, E_{dju} ; M)
\]

should be \((-1)^{q} d(j\tilde{u})\) for \( u \in \tilde{B}_p \). Then we have

\[
d^1 i(u) = \sum_{0 \leq j \leq p} (-1)^{qi} i(d(j\tilde{u})) d^1(lq)(u),
\]

where

\[
i(u) : H_{p+q}(E_{u'}, E_{u} ; M) \rightarrow E^1_{p,q} C_*(\pi ; M)
\]

is the canonical injection as earlier.

6. Remarks on various types of fibrations

6.1. Definition. A simplicial morphism \( \pi : E \rightarrow B \) is a weak fibration if \( \pi \) is surjective and whenever

\[
\begin{array}{ccc}
E_{f_1} & \xrightarrow{f} & E_f & \xrightarrow{f} & E \\
\pi_{f_1} \downarrow & & \pi_f \downarrow & & \pi \\
A' & \xrightarrow{i} & A & \xrightarrow{f} & B
\end{array}
\]
is a commutative diagram in $\Delta($sets$)$ such that the squares are Cartesian and $i$ is a homotopy equivalence, then $\tilde{r}$ is a homotopy equivalence.

6.2. Properties of weak fibrations.

(1) A weak fibration is an $R$-fibration for any ring $R$.

(2) A morphism to the base of a weak fibration induces a weak fibration over its domain.

(3) Composites of weak fibrations are weak fibrations.

(4) The projection of a product to one of its factors is a weak fibration.

(5) Any fibration (i.e. Kan fibration [9, 24]) is a weak fibration.

Recall that the definition of fibration in $\Delta($sets$)$ is just the literal translation to this category of one of Serre's alternative definitions of fibration in the category of spaces and maps; see [25]. The advantage of weak fibration over fibration is the fact that they enjoy 6.2(4), which is not shared by fibrations unless one restricts oneself to simplicial sets satisfying the extension condition.

6.3. Remark. For a pointed weak fibration there is a homotopy sequence of the usual type.

For this, note that if

$$
\begin{array}{ccc}
E' & \xrightarrow{g} & E \\
\downarrow{h} & & \downarrow{\pi} \\
B' & \xrightarrow{f} & B
\end{array}
$$

is a Cartesian square of pointed simplicial sets such that either $\pi$ or $f$ is a fibration, then there is an exact homotopy sequence

$$
\ldots \rightarrow \pi_q(E') \rightarrow \pi_q(B') \oplus \pi_q(E) \rightarrow \pi_q(B) \rightarrow \pi_{q-1}(E') \rightarrow \ldots
$$

of Mayer–Vietoris type. Assuming that $B$ is connected, let $f$ be the canonical fibration over $B$ with $B'$ contractible, and the sequence becomes

$$
\ldots \rightarrow \pi_q(E') \rightarrow \pi_q(E) \rightarrow \pi_q(B) \rightarrow \pi_{q-1}(E') \rightarrow \ldots .
$$

If $\pi$ is a weak fibration with fibre $F$, there is a canonical morphism $i : F \rightarrow E'$ which is a homotopy equivalence, and the exact sequence above becomes the homotopy sequence of the weak fibration $\pi$. 
IV. The collapsing theorem

The object of this part is to prove the main theorem of the paper concerning the collapsing of the induced fibration spectral sequence for the fibration $BU \to BG$. This involves a deeper analysis of certain homotopy properties of the functors $B$ and $\Omega$; in particular, their behavior relative to contractible algebras and coalgebras.

The main preliminaries needed for the collapsing theorem are two splitting theorems: Theorem 3.6, which says that for certain injective morphisms $i : D \to E$ of coalgebras there is a morphism of algebras $g : \Omega(E) \to \Omega(D)$ with $g\Omega(i) = \Omega(D)$, and Theorem 2.5, which says that for certain surjective morphisms $\pi : A \to A''$ of algebras there is a morphism of coalgebras $s : B(A'') \to B(A)$ with $B(\pi)s = B(A'')$.

From the algebra splitting theorem we will be able to construct a morphism of algebras

$$\psi(A, A') : \Omega B(A \otimes A') \to \Omega B(A) \otimes \Omega B(A')$$

which is functorial in $A$ and $A'$; see 5.5.

1. Cones and cocones

Recall that a projective in the category $\mathcal{C}(R)$ is an object $P$ such that $P_n$ is a projective $R$-module for all integers $n$ and such that there exists $c \in [P, P]$ with $c^2 = 0$ and $Dc = P$, i.e., $d(P)c + c d(P) = P$. Such a morphism $c$ of degree $+1$ is called a conical contraction. Similarly, an injective is an object which is injective in all degrees and has a conical contraction. With respect to $R$-split sequences, the objects in $\mathcal{C}(R)$ with conical contractions are both relative injectives and relative projectives. This results from three sets of adjoint pairs considered in [14, §8] which we review here.

1.1. Notation. Let $\mathcal{A}$ be an abelian category. Let $J : \text{Gr}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ denote the inclusion, where $d(J(A)) = 0$ on the graded object $A$ in $\mathcal{C}(\mathcal{A})$. For a differential object $X$ in $\mathcal{C}(\mathcal{A})$ we have the following exact sequence over $\text{Gr}(\mathcal{A})$:

$$0 \to Z(X) \to X \xrightarrow{d(x)} X \to Z'(X) \to 0.$$ 

1.2. Remark. There are adjoint relations $Z' \dashv J$ and $J \dashv Z$. In other words, any morphism $X \to N$ in $\mathcal{C}(\mathcal{A})$, where $d(N) = 0$, factors uniquely as

$$X \to Z'(X) \to N,$$

and similarly any morphism $M \to X$ in $\mathcal{C}(\mathcal{A})$, where $d(M) = 0$, factors uniquely as

$$M \to Z(X) \to X.$$

The functor $J$ is exact and faithful.
1.3. Notations. Let \( R : \mathcal{C}(\mathcal{A}) \to \text{Gr}(\mathcal{A}) \) denote the structure reduction functor which assigns to a differential object \( X \) its underlying graded object. The cone functor
\[
C : \text{Gr}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})
\]
and the cocone functor
\[
C' : \text{Gr}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})
\]
are defined by the relations
\[
C(A)_n = A_n \oplus A_{n-1} \quad \text{and} \quad C'(A)_n = A_n \oplus A_{n+1}
\]
as graded objects. The morphisms
\[
dC'(A)_n : C'(A)_n = A_n \oplus A_{n+1} \to A_{n-1} \oplus A_n = C(A)_{n-1}
\]
is defined using the identity \( A_n \to A_n \) and zeros between other factors, and the same for \( dC(A)_n \). In terms of elements \( dC'(A)_n(x, y) = (0, x) \) for \( x \in A_n, y \in A_{n+1} \).

1.4. Remark. There are adjoint relations \( C' \dashv R \) and \( R \dashv C \). The functors \( C', R \) and \( C \) are exact and faithful.

For \( \mathcal{C}(\mathcal{A}) = \mathcal{C}(R) \) we can make explicit in terms of elements the adjunction morphisms \( (\alpha', \beta') : C' \to R \) and \( (\alpha, \beta) : R \to C \). The morphism \( \alpha'(X) : C'R(X) \to X \) is given by
\[
\alpha'(X)_n(x, y) = x + d(X)_{n+1}(y)
\]
and \( \beta'(A) : A \to RC'(A) \) by
\[
\beta'(A)_n(y) = (y, 0).
\]
For \( C(A)_n = A_n \oplus A_{n-1} \), with \( d(C(A))_n(x, y) = (y, 0) \), the morphism \( \beta(X) : X \to CR(X) \) is given by
\[
\beta(X)_n(x) = (x, dx) \quad \text{for} \quad x \in X_n
\]
and \( \alpha(A) : RC(A) \to A \) by
\[
\alpha(A)_n(x, y) = x.
\]

1.5. Remarks. In \( \text{Gr}(\mathcal{A}) \) we have the split projective class \( \mathcal{P}_s \) and split injective class \( \mathcal{I}_s \). By I, 1.3, we use \( C' \dashv R : \mathcal{C}(\mathcal{A}) \to \text{Gr}(\mathcal{A}) \) to reflect \( \mathcal{P}_s \) into a projective class in \( \mathcal{C}(\mathcal{A}) \), called the contractible projective class. The projectives are factors of co-cones \( C'(A) \), and the epimorphisms are morphisms \( u : X \to Y \) such that \( R(u) \) has a right inverse in \( \text{Gr}(\mathcal{A}) \). By I, 1.3, we use \( R \dashv C : \text{Gr}(\mathcal{A}) \to \mathcal{C}(\mathcal{A}) \) to reflect \( \mathcal{I}_s \) into an injective class in \( \mathcal{C}(\mathcal{A}) \), called the contractible injective class. The injectives are factors of cones \( C(A) \) and the monomorphisms are morphisms \( u : X \to Y \) such that \( R(u) \) has a left inverse in \( \text{Gr}(\mathcal{A}) \).
The functor \( CR : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A}) \) is the cone functor of \( \mathcal{C}(\mathcal{A}) \), and \( C'R : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A}) \) is the cocone functor of \( \mathcal{C}(\mathcal{A}) \). The reader can show that this definition of the cone functor is isomorphic to the mapping cylinder definition. For \( s \) the suspension functor, \( C's \) and \( C \) are naturally isomorphic. Hence \( X \) in \( \mathcal{C}(\mathcal{A}) \) is a contractible projective if and only if it is a contractible injective. Note that an object in \( \mathcal{C}(\mathcal{A}) \) is contractible if and only if it is isomorphic to a cone, or equivalently, a cocone. Indeed, if \( f \) is any contraction of \( X \), i.e. \( Df = X \), then \( c = fd(X)f \) is a conical contraction of \( X \), i.e. \( Dc = X \) and \( c^2 = 0 \) and \( X \) is a cone \( X = CZX \). Moreover, \( cd(X)c = c \). Observe that this also implies that a factor of a contractible object is a contractible object. Thus the contractible objects are both the projectives for the contractible projective class and the injectives for the contractible injective class.

1.6. Remarks. Putting together the adjoint relations of 1.2 and 1.4, we obtain two adjoint relations
\[
J \circ C' \to (Z, R) : \mathcal{C}(\mathcal{A}) \to \text{Gr}(\mathcal{A}) \times \text{Gr}(\mathcal{A}),
\]
\[
(Z', R) \to J \circ C : \text{Gr}(\mathcal{A}) \times \text{Gr}(\mathcal{A}) \to \mathcal{C}(\mathcal{A}).
\]
If \( \mathcal{A} \) has sufficiently many projectives or injectives, then so does \( \text{Gr}(\mathcal{A}) \), and we can speak of the absolute projective or injective classes in \( \text{Gr}(\mathcal{A}) \times \text{Gr}(\mathcal{A}) \). The hyperhomology projective class in \( \mathcal{C}(\mathcal{A}) \) is the reflection of the absolute projective class in \( \text{Gr}(\mathcal{A}) \times \text{Gr}(\mathcal{A}) \) by the adjoint pair \( J \circ C' \to (Z, R) \) using I, 1.3. The hyperhomology injective class in \( \mathcal{C}(\mathcal{A}) \) is the reflection of the absolute injective class in \( \text{Gr}(\mathcal{A}) \times \text{Gr}(\mathcal{A}) \) by the adjoint pair \( (Z', R) \to J \circ C \) using I, 1.3.

Moreover, for an arbitrary abelian category \( \mathcal{A} \) the first adjoint relation \( J \circ C' \to (Z, R) \) gives rise to the split hyperhomology projective class in \( \mathcal{C}(\mathcal{A}) \) as the reflection of the split projective class in \( \text{Gr}(\mathcal{A}) \times \text{Gr}(\mathcal{A}) \) using I, 1.3. Also, the second adjoint relation \( (Z', R) \to J \circ C \) gives rise to the split hyperhomology injective class in \( \mathcal{C}(\mathcal{A}) \) as the reflection of the split injective class in \( \text{Gr}(\mathcal{A}) \times \text{Gr}(\mathcal{A}) \) using I, 1.3. In particular, the objects in \( \mathcal{C}(\mathcal{A}) \) isomorphic to an object of the form \( X \oplus Y \), where \( X \) is contractible and \( d(Y) = 0 \), are exactly the split hyperhomology projectives and also the split hyperhomology injectives.

2. Splitting theorem for algebras

2.1. Definition. A supplemented algebra \( A \) is contractible provided \( I(A) \) is contractible in \( \mathcal{C}(R) \).

In other words, \( I(A) \) is isomorphic to the cone \( C(Z(I(A))) \) as a differential module, where the isomorphism may be constructed from the contracting homotopy. This condition is equivalent to \( e(A) : A \to R \) being a homotopy equivalence in \( \mathcal{C}(R) \).
2.2. Proposition. A supplemented algebra $A$ is contractible if and only if $B(A)$ is isomorphic to $T'(C(sZ(I(A))))$ in $\text{Coalg}(R)$. In particular this implies that $H(B(A)) = R$.

Proof. If $A$ is contractible, then in $\mathcal{C}_c(R)$ the natural morphism $i: sl(A) \to I(B(A))$ has a left inverse $f$ because $sl(A)$ is contractible and $i$ is a monomorphism in the contractible injective class. Now $f : I(B(A)) \to sl(A)$ induces a unique morphism $f' : B(A) \to T'(sl(A))$ in $\text{Coalg}(R)$ which composed with the projection $T'(si(A)) \to sl(A)$ is $f$. Filter both $B(A)$ and $T'(sl(A))$ by their coalgebra filtrations, which in this case is just the tensor degree, and observe that $E^{0}(f')$ is an isomorphism using its construction from $f$. Since the coalgebras are connected, the coalgebra filtrations are bicomplete, and hence $f'$ is an isomorphism.

Conversely, if $B(A)$ is isomorphic to $T'(C(X))$, then the primitive element differential module $PB(A) = sl(A)$ is isomorphic to the cone $C(X) = PT'(C(X))$, and hence $I(A)$ is contractible. This proves the proposition. ⊓⊔

2.3. Remark. Since $I \to T': \mathcal{C}_c(R) \to \text{Coalg}(R)$ is an adjoint pair, we see that the contractible injective class of $\mathcal{C}_c(R)$, considered in 1.5, is reflected into $\text{Coalg}(R)$ by $I$, 1.3 to what we call the $R$-split injective class of $\text{Coalg}(R)$. The injectives are factors of $T'(C(X))$, and the monomorphisms are morphisms which are split monomorphisms in $\mathcal{C}_c(R)$.

Recall that the kernel of a morphism $j : A \to A''$ of supplemented algebras is the unique morphism $i : A' \to A$ of supplemented algebras such that $I(i)$ is the kernel of $I(j)$ in $\mathcal{C}_c(R)$.

2.4. Definition. A morphism $j : A \to A''$ of supplemented algebras is a fibraticln provided it is a split epimorphism when viewed in the category $\text{Gr}_c(R)$. The fibre of $j$ is its kernel $i : A' \to A$ (or simply $A'$ by abuse of language).

The next theorem is the basic splitting result for fibrations of algebras. The decomposition is a product representation of $B(A)$ for a fibration $j : A \to A''$ in terms of $B(A'')$ under suitable hypothesis.

2.5. Theorem. Let $j : A \to A''$ be a fibration in $\text{Alg}(R)$ with fibre $i : A' \to A$ such that $A'$ is contractible. Then there is a morphism of connected coalgebras $g : B(A) \to B(A')$ such that $gB(i) = B(A')$ and

$$B(j) \circ g : B(A) \to B(A'') \cong B(A')$$

is an isomorphism.

Proof. By 2.3 the coalgebra $B(A')$ is an injective coalgebra relative to $R$-split monomorphisms in $\text{Coalg}(R)$. Since $B(i)$ is an $R$-split monomorphism, there exist a $g : B(A) \to B(A')$ with $gB(i) = B(A')$. 

For $B(j) \tau g : B(A) \to B(A\pi B(A')$, the morphism $E^0(B(j) \pi g)$ relative to the coalgebra filtrations is essentially

$$T'(sl(A)) \to T'(sl(A')) \pi T'(sl(A')),$$

the isomorphism induced by the splitting in $C_+(R)$ of

$$0 \to I(A') \to I(A) \to I(A'') \to 0.$$

This splitting arises from the fact that $I(A')$ is an injective for the graded split injective class in $C_+(R)$. Since $E^0(B(j) \pi g)$ is an isomorphism, $B(j) \pi g$ is also an isomorphism, and this proves the theorem. ☐

3. Splitting theorem for coalgebras

3.1. Definition. A connected coalgebra $C$ is contractible provided the algebra $\Omega(C)$ is contractible.

So by 2.1, this means that $I\Omega(C)$ is contractible in $C_+(R)$.

3.2. Proposition. If $D$ is a simply connected coalgebra such that $I(D)$ is contractible in $C_+(R)$, then $\Omega(D)$ is isomorphic to $T(C'ZI(D))$ in Alg(R), and so $D$ is contractible.

Proof. Since $I(D)$ is contractible in $C_+(R)$, there is an isomorphism of degree $+1$, denoted $\lambda : C'(X) \to I(D)$, where $X = ZI(D)$, and $\lambda$ is induced by the morphism $\lambda I(D) : X \to I(D)$ of degree $+1$ in $\text{Gr}(R)$. Thus $\tau(D)\lambda I(D) : X \to I\Omega(D)$ is a morphism in $\text{Gr}(R)$ and induces $\tilde{\lambda} : C'(X) \to I\Omega(D)$ in $C_+(R)$. By the universal property of the tensor algebra, $\tilde{\lambda} : TC'(X) \to \Omega(D)$ in Alg($R$). Filtering by powers of the augmentation ideals, one sees at once that $E^0(\tilde{\lambda})$ is an isomorphism. Since the coalgebra $D$ is simply connected, the algebras are connected, and this implies that $\tilde{\lambda}$ is an isomorphism.

Finally, it is evident that $TC'(X)$ is contractible, and thus $D$ is contractible. This proves the proposition. ☐

3.3. Example. Let $F$ be a free group on a countable set of generators, $F'$ its commutator subgroup, $j : F \to F'$ an isomorphism, and $i : F' \to F$ the natural inclusion. For $u = ij$ let $G$ be the colimit of the countable system.

$$F \xrightarrow{u} F \xrightarrow{u} \ldots \xrightarrow{u} F \xrightarrow{u} \ldots$$

The group $G$ is its own commutator subgroup and has cohomological dimension 2. The coalgebra $B(R[G])$ has trivial homology, $I(B(R[G]))$ is contractible to zero in $C_+(R)$, but $B(R[G])$ is not a contractible coalgebra since

$$\alpha(R[G]) : \Omega B(R[G]) \to R[G]$$
is a homotopy equivalence in $C_+(R)$. Thus the simple connectivity hypothesis is necessary in 3.2.

In the nonsimply connected case we have the following result.

3.4. Proposition. Let $D$ be a connected coalgebra such that $I(D)$ is a projective in $Gr_+(R)$. Then $D$ is contractible if and only if $H(I(D)) = 0$ and $H_0(\Omega(D)) = R$.

Proof. If $D$ is contractible, this means that $I\Omega(D)$ is contractible, and so $H_0(\Omega(D)) = R$. Since $D \to B\Omega(D)$ is a homotopy equivalence by II, 3.5, and $H(B\Omega(D)) = R$ by 2.2, it follows that $H(D) = R$ and $H(I(D)) = 0$.

Conversely, since $H_0(\Omega(D)) = R$, the local coefficients are trivial in the Serre spectral sequence for $E_1^J(D)$ and $E_1^J = D \otimes H(\Omega(D))$ because, in addition, $D$ is projective in $Gr_+(R)$. Since $H(I(D)) = 0$ and $I(D)$ is projective in $Gr_+(R)$, $I(D)$ is contractible, and so the natural inclusion $\Omega(D) \to E_1^J(D)$ is a homology equivalence because the $E^2$ terms map isomorphically. Now $H(E_1^J(D)) = R$, and so $H(\Omega(D)) = R$ and $H(I\Omega(D)) = 0$. Again $I\Omega(D)$, being a direct sum of tensor products of $s^{-1}I(D)$ in $Gr_+(R)$, is projective in $Gr_+(R)$. Thus $I\Omega(D)$ is contractible, which by definition means that $D$ is contractible. This proves the proposition. □

Recall that the cokernel of a morphism $i : D' \to D$ of connected coalgebras is the unique morphism $j : D \to D''$ of connected coalgebras such that $I(j)$ is the cokernel of $I(i)$ in $C_+(R)$.

3.5. Definition. A morphism $i : D' \to D$ of connected coalgebras is a cofibration provided it is a split monomorphism when viewed in the category $Gr_+(R)$. The cofibre of $i$ is its cokernel $j : D \to D''$ (or simply $D''$ by abuse of language).

The following is the first version of the splitting theorem for coalgebras. Note that this involves no assumption of simple connectivity.

3.6. Theorem. The following are equivalent for a connected coalgebra $D$:

1. $D$ is contractible,
2. $d_1 : I\Omega(D)_1 \to I\Omega(D)_0$ is a split epimorphism in $Gr(R)$, and $\Omega(N)$ is a projective in $Alg(R)$ relative to fibrations,
3. for a cofibration $i : D \to E$ in $Coalg(R)$ there exists a morphism $g : \Omega(E) \to \Omega(D)$ such that $g\Omega(i) = \Omega(D)$.

Proof. For (1) $\Rightarrow$ (2), observe that the first assertion in (2) is immediate from the fact that $I\Omega(D)$ is contractible and isomorphic to a cocone in $C_+(R)$. The natural injection $I\Omega(D) \to \Omega(D)$ prolongs to a fibration $f : T(I\Omega(D)) \to \Omega(D)$ in $Alg(R)$ with fibre $\Lambda : A \to T(I\Omega(D))$, where $A$ is contractible. By 2.5 there exists $g : BT(I\Omega(D)) \to B(A)$ such that $gB(i) = B(A)$.
and \( B(f) \cdot g : BT(i_\Omega(D)) \to B\Omega(D) \pi B(A) \) is an isomorphism. Thus there exists \( h : \Omega(D) \to T(i_\Omega(D)) \) with \( jh = \Omega(D) \) and \( \Omega(D) \) is a retract of the projective \( T(i_\Omega(D)) \) hence a projective.

Conversely, the first part of (2) implies that there is a cocone \( X \) and an \( R \)-split epimorphism \( f' : X \to i_\Omega(D) \) in \( C_\bullet(R) \) which prolongs to a fibration \( f : T(X) \to \Omega(D) \) in \( Alg(R) \). The other part of (2) implies that there exists an algebra morphism \( f : \Omega(D) \to T(X) \) with \( if = \Omega(D) \). Hence \( I\Omega(D) \) is contractible, being a factor of \( IT(X) \) in \( C_\bullet(R) \), and (1) holds.

For (1) \( \Rightarrow \) (3), consider \( i : D \to E \), a cofibration in \( Coalg(R) \). Then \( B\Omega(i) : B\Omega(D) \to B\Omega(E) \) is a cofibration, and by 2.2, \( B\Omega(D) \) is isomorphic to \( T' \) applied to a cone. This means that \( B\Omega(D) \) is an injective relative to cofibrations, and there exists \( h : B\Omega(E) \to B\Omega(D) \) such that

\[
h(B\Omega(i)) = B\Omega(D).
\]

For \( g = \alpha(\Omega(E)) \Omega(h\beta(E)) \) we have

\[
g\Omega(i) = \Omega(D),
\]

and (3) holds.

Conversely, let \( f : D \to T'(X) \) be a cofibration with \( X \) a cone in \( C_\bullet(R) \). By (3), \( \Omega(f) \) is a split monomorphism in \( Alg(R) \), and \( I\Omega(D) \) is a factor of \( I\Omega T'(X) \) in \( C_\bullet(R) \). Since \( I\Omega T'(X) \) is contractible, \( I\Omega(D) \) is contractible, and this proves the theorem. \( \square \)

Now we consider the second splitting theorem, valid only for simply connected coalgebras.

3.7. Theorem. Let \( i : D' \to D \) be a cofibration of simply connected coalgebras with cofibre \( j : D \to D'' \) and \( D'' \) contractible. Then there exists \( g : \Omega(D'') \to \Omega(D) \) such that \( \Omega(j)g = \Omega(D') \). Such a morphism has the property that

\[
g \cdot \Omega(i) : \Omega(D'') \cup \Omega(D') \to \Omega(D)
\]

is an isomorphism of algebras.

Proof. By 3.6, the algebra \( \Omega(D'') \) is a projective in \( Alg(R) \) relative to fibrations. Since \( \Omega(j) \) is a fibration, there exists a morphism \( g : \Omega(D'') \to \Omega(D) \) such that \( \Omega(j)g = \Omega(D'') \).

For the second statement observe that

\[
g \cdot \Omega(i) : \Omega(D'') \cup \Omega(D') \to \Omega(D)
\]

preserves the algebra filtration, i.e., the filtration by powers of the augmentation ideal. Up to natural identifications, \( E^0(g \cdot \Omega(i)) \) is just the isomorphism

\[
T(s^{-1}(D'')) \cup T(s^{-1}(D')) \to T(s^{-1}(D))
\]
induced by a splitting
\[ 0 \to I(D') \to I(D) \to I(D'') \to 0 \]
in \( C_+^\times(R) \). The simple connectivity hypothesis implies that the above algebras are connected, and so the algebra filtrations are bicomplete, i.e. complete and cocomplete. Hence \( g \circ \Omega(j) \) is an isomorphism, and this proves the theorem. \( \square \)

4. Further properties of cofibrations of coalgebras

For a cofibration of path-connected spaces \( X \to Y \to Z \), recall that if \( X \) is contractible, then \( Y \to Z \) is a homotopy equivalence, and if \( Z \) is contractible and \( X \) and \( Y \) are simply connected, then \( X \to Y \) is a homotopy equivalence. In this section we study the related results for coalgebras.

4.1. Definition. A deformation cofibration \( i : D' \to D \) of connected coalgebras is a cofibration such that \( I\Omega(i) : I\Omega(D') \to I\Omega(D) \) is a homotopy equivalence in \( C_+^\times(R) \) and there exists \( g : \Omega(D) \to \Omega(D') \) in \( \text{Alg}(R) \) with \( g\Omega(i) = \Omega(D') \).

4.2. Proposition. Let \( i : D' \to D \) be a cofibration of simply connected coalgebras such that the cofibre \( D'' \) is contractible. Then \( i : D' \to D \) is a deformation cofibration.

Proof. This is immediate from 3.7. \( \square \)

4.3. Example. For \( n \geq 2 \), let \( A \) denote the bouquet \( S^1 \vee S^n \) of a 1-sphere and an \( n \)-sphere together with natural inclusions \( \alpha : S^1 \to A, i : S^n \to A \). Let \( \alpha \) and \( i \) denote their corresponding classes in \( \pi_1(A) \) and \( \pi_n(A) \), respectively. Let \( X \) denote the space obtained by attaching an \((n + 1)\)-cell to \( A \) by \( 2i - i^2 \), where \( i^2 \) is the element of \( \pi_n(A) \) obtained from \( i \) by acting with \( \alpha \).

For the pair \((X, S^1)\) observe that
\[ H_q(X, S^1; Z) = \begin{cases} 0 & \text{for } q < n, \\ \pi_q(X, S^1) & \text{for } q = n, \\ \pi_n(X, S^1) \cong \mathbb{Z}[\frac{1}{2}] & \text{for } q > n. \end{cases} \]
Using associative loop spaces, we obtain two algebras \( \Lambda = C_*(\Omega(S^1)) \) and \( \Gamma = C_*(\Omega(X)) \) and a natural injection \( f : \Lambda \to \Gamma \) over the integers. Now \( B(f) : B(\Lambda) \to B(\Gamma) \) is a cofibration of coalgebras, and the cofibre \( D \) has the same homotopy type as \( C_*(X/S^1) \). Thus \( D \) is contractible. However, \( B(f) \) is not a deformation cofibration since
\[ H_q(\Omega B \Lambda) = H_q(\Lambda) = 0 \]
for \( q \neq 0 \) and \( \pi_n(X) = \pi_n(X, S^1) \) injects into \( H_{n-1}(\Omega B(\Gamma)) = H_{n-1}(\Gamma) \). Further note that the space \( X/S^1 \) is contractible.

4.4. Definition. A morphism \( j : D \to D'' \) of coalgebras is a surjective deformation provided \( I\Omega(j) : I\Omega(D) \to I\Omega(D'') \) is an \( R \)-split epimorphism which is a homotopy equivalence in \( C_+^\times(R) \).
4.5. Definition. Let \( i : D' \to D \) be a cofibration of coalgebras. The \( i \)-cofibration filtration on \( D \) is the positive filtration \( F_pD \), where

\[
(F_pD)_n = \begin{cases} 
D_n & \text{for } n \leq p, \\
i(D'_n) & \text{for } n > p. 
\end{cases}
\]

Observe that \( F_pD \) is a coalgebra for all \( p \) and \( F_pD \to D \) is a cofibration. The \( i \)-cofibration filtration on \( D \) coming from \( i : R \to D \) is just the skeleton filtration. The \( i \)-cofibration filtration on \( D \) induces a filtration on \( \Omega(D) \) by setting

\[
F_p\Omega(D) = \Omega(F_pD),
\]
called the \( i \)-filtration on \( \Omega(D) \).

If \( j : D \to D'' \) is the cofibre of a cofibration \( i : D' \to D \), where \( D \) has the \( i \)-cofibration filtration, and if \( D'' \) has the skeleton filtration, then \( j \) preserves the filtrations, and we can identify \( E^0(D) \) with \( E^0(D'') \oplus D' \) in \( \mathcal{C}_+(R) \) canonically. If the natural morphism \( E^0(D'') \to E^0(D) \) in \( \mathcal{C}_+(R) \) is a morphism of coalgebras, then the cofibration is said to split. Assuming \( H(D') \) is projective, we have

\[
E^1(D) = E^1(D'') \oplus H(D')
\]
in \( \mathcal{C}_+(R) \) which as a coalgebra we can identify \( E^1(D'') \) with \( D'' \). The cofibration is homologically split provided \( D'' \to E^1(D) \) is a morphism of coalgebras.

The proofs of the following proposition and its corollaries are left to the reader.

4.6. Proposition. Let \( i : D' \to D \) be a cofibration with cofibre \( j : D \to D'' \), and give \( \Omega(D) \) the \( i \)-filtration and \( \Omega(D'') \) the \( \eta \)-filtration. Then

\[
E^0\Omega(D) = E^0\Omega(D'') \oplus \Omega(D'),
\]
and the differential on \( E^0\Omega(D'') \) is zero. In addition, if \( H(D') \) and \( H(\Omega D') \) are projectives in \( \text{Gr}(R) \), and \( i \) is homologically split, then

\[
E^1\Omega(D) = \Omega(D'') \oplus H\Omega(D').
\]

4.7. Corollary. If \( i : D' \to D \) is a cofibration with cofibre \( j : D \to D'' \), \( H(D') \) and \( H\Omega(D') \) are projective in \( \text{Gr}(R) \), and the cofibration is homologically split, then

\[
E^2\Omega(D) = H(\Omega(D'')) \oplus H(\Omega(D')).
\]

4.8. Corollary. If \( i : D' \to D \) is a cofibration with cofibre \( j : D \to D'' \), and \( D' \) is contractible, then \( j \) is a surjective deformation.

4.9. Corollary. If \( i : D' \to D \) is a cofibration with cofibre \( j : D \to D'' \) which is homologically split, \( H(D') \) and \( H\Omega(D') \) are projective in \( \text{Gr}(R) \), and \( D'' \) is contractible, then \( i \) is a deformation cofibration.
5. Quasi-Hopf algebras and quasi-commutative coalgebras

5.1. Definition. A quasi-Hopf algebra \( A \) is a supplemented algebra \( A \) together with a morphism \( \Delta^*(A) : A \to A \otimes A \) of algebras such that
\[
(\varepsilon(A) \otimes A) \Delta^*(A) = A = (A \otimes \varepsilon(A)) \Delta^*(A).
\]
For two quasi-Hopf algebras \( A \) and \( A' \), a morphism \( f : A \to A' \) is a morphism of supplemented algebras such that \((f \otimes f) \Delta^*(A) = \Delta^*(A')f\).

Composition of morphisms of quasi-Hopf algebras is induced by composition of morphisms of the underlying supplemented algebras. Thus we can speak of \( \text{Alg}^*(R) \), the category of quasi-Hopf algebras. Since a quasi-Hopf algebra \( A \) is a Hopf algebra if and only if \( \Delta^*(A) \) is associative, the category of Hopf algebras forms a full subcategory of \( \text{Alg}^*(R) \).

5.2. Definition. A quasi-commutative coalgebra \( C \) is a supplemented coalgebra \( C \) together with a quasi-Hopf algebra structure \( \Delta^*\Omega(C) \) on the algebra \( \Omega(C) \). A morphism \( g : C \to C' \) of quasi-commutative coalgebras is a morphism of the underlying coalgebras such that \( \Omega(g) : \Omega(C) \to \Omega(C') \) is a morphism of quasi-Hopf algebras.

The adjoint of the quasi-commutated structure morphism
\[
\Delta^*\Omega(C) : \Omega(C) \to \Omega(C) \otimes \Omega(C)
\]
is a morphism
\[
\psi_C : C \to B(\Omega(C) \otimes \Omega(C))
\]
of coalgebras. The counit conditions in 5.1 take the form
\[
B(\varepsilon\Omega(C) \otimes \Omega(C))\psi_C = id_C = B(\Omega(C) \otimes \varepsilon\Omega(C))\psi_C.
\]
Composition of morphisms of quasi-commutative coalgebras is induced by composition of the morphisms of the underlying supplemented coalgebras. Thus we can speak of \( \text{Coalg}^*(R) \), the category of quasi-commutative coalgebras. The functor \( \Omega : \text{Coalg}(R) \to \text{Alg}(R) \) extends to a functor \( \Omega : \text{Coalg}^*(R) \to \text{Alg}^*(R) \) from the character of the definitions. The main problem of this section is to determine to what extent a functor \( B : \text{Alg}^*(R) \to \text{Coalg}^*(R) \) adjoint to \( \Omega \) can be defined.

In order to see that any commutative coalgebra has a natural quasi-commutated structure and that \( B(A) \) has a quasi-commutative structure for a quasi-Hopf algebra \( A \), we need the following natural morphisms related to the functors \( B \) and \( \Omega \), defined using the canonical twisting morphisms \( \tau(A) : B(A) \to A \) and \( \tau(C) : C \to \Omega(C) \).

5.3. Definition. The shuffle morphism \( \gamma : B(A) \otimes B(A') \to B(A \otimes A') \) is defined by the relation
\[
\tau(A \otimes A')\gamma = \tau(A) \otimes \eta(A') + \eta(A) \otimes \tau(A').
\]
The shuffle morphism $\gamma' : \Omega(C \otimes C') \to \Omega(C) \otimes \Omega(C')$ is defined by the relation.

$$\gamma'(\tau(C \otimes C')) = \tau(C) \otimes \eta(C') + \eta(C) \otimes \tau(C').$$

The naturality of the shuffle morphisms

$$\gamma : B(A) \otimes B(A') \to B(A \otimes A'), \quad \gamma' : \Omega(C \otimes C') \to \Omega(C) \otimes \Omega(C')$$

results immediately from the naturality properties of $\tau(A)$ and $\tau(C)$ considered in II, 3.5. These shuffle morphisms extend to $n$ variables yielding a cofibration of coalgebras with contractible cofibre

$$\gamma : B(A(1)) \otimes \ldots \otimes B(A(n)) \to B(A(1) \otimes \ldots \otimes A(n))$$

and a fibration of algebras with contractible fibre

$$\gamma' : \Omega(C(1) \otimes \ldots \otimes C(n)) \to \Omega(C(1)) \otimes \ldots \otimes \Omega(C(n)).$$

Recall that a coalgebra $C$ is commutative if and only if $\Delta(C) : C \to C \otimes C$ is a morphism of coalgebras.

5.4. Definition. The quasi-commuted structure $\Delta\Omega(C) : \Omega(C) \to \Omega(C) \otimes \Omega(C)$ on a commutative coalgebra $C$ is the morphism adjoint to the composite

$$C \xrightarrow{\Delta(C)} C \otimes C \xrightarrow{(\beta(C) \otimes \beta(C))} B\Omega(C) \otimes B\Omega(C) \xrightarrow{\tau} B(\Omega(C) \otimes \Omega(C)).$$

Hence $\Omega$ is a functor from the category of commutative coalgebras to the category of cocommutative Hopf algebras. In particular, $\Delta\Omega(C)$ is a quasi-commuted structure on $C$.

In order to study quasi-commuted structures on $B(A)$ for a quasi-Hopf algebra $A$ we need the following proposition.

5.5. Proposition. There is a functorial morphism

$$\psi = \psi(A, A') : \Omega B(A \otimes A') \to \Omega B(A) \otimes \Omega B(A')$$

such that $H_\phi(\psi)$ is an isomorphism, $\psi(A, R) = \psi(R, A)$ is the identity on $\Omega B(A)$, and

$$\alpha(A \otimes A') = [\alpha(A) \otimes \alpha(A')] \psi(A, A').$$

Proof. Let $N(A, A')$ be the kernel in the category of algebras of

$$\alpha(A) \otimes \alpha(A') : \Omega B(A) \otimes \Omega B(A') \to A \otimes A'.$$

If $N(A) \to \Omega B(A)$ is the algebra kernel of $\alpha(A) : \Omega B(A) \to A$, then

$$IN(A, A') = [A \otimes IN(A')] \otimes [IN(A) \otimes !N(A')] \otimes [IN(A) \otimes A'].$$
By II, 4.4, we have a conical contraction \( c(A) \) of \( IN(A) \) depending functorially on \( A \), and hence
\[
c(A, A') = [A \otimes c(A')] \oplus [c(A) \otimes N(A')] \oplus [c(A) \otimes A']
\]
is a conical contraction on \( IN(A \otimes A') \) depending functorially on \( A \) and \( A' \).

By 2.5 there is a morphism of coalgebras
\[
\sigma = \sigma(A, A') : B(A \otimes A') \rightarrow B(\Omega B(A) \otimes \Omega B(A'))
\]
depending functorially on \( A \) and \( A' \) such that
\[
B(\alpha(A) \otimes \alpha(A')) \circ \sigma(A, A') = B(A \otimes A').
\]
Clearly \( H_*(\sigma) \) is an isomorphism, and \( \sigma(A, R) = \sigma(R, A) \) is just \( B(\beta(A)) \). Finally we define \( \psi = \psi(A, A') \) as the composite
\[
\begin{array}{c}
\Omega B(A \otimes A') \\
\downarrow \psi(A, A') \\
\downarrow \alpha(\Omega B(A) \otimes \Omega B(A'))
\end{array}
\begin{array}{c}
\Omega B(\Omega B(A) \otimes \Omega B(A')) \\
\downarrow \alpha(\Omega B(A) \otimes \Omega B(A'))
\end{array}
\]
or in other words the adjoint of \( \sigma = \sigma(A, A') \). From the properties of \( \sigma \), we deduce the first two properties of \( \psi(A, A') \), and the third follows from the commutative diagram
\[
\begin{array}{c}
\Omega B(\Omega B(A) \otimes \Omega B(A')) \\
\downarrow \alpha(\Omega B(A) \otimes \Omega B(A'))
\end{array}
\begin{array}{c}
\xrightarrow{\Omega B(\alpha(A) \otimes \alpha(A'))} \\
\Omega B(A \otimes A') \\
\downarrow \alpha(A \otimes A')
\end{array}
\]
which remains commutative after \( \Omega(\sigma) \) is added. This proves the proposition. \( \square \)

5.6. **Theorem.** There exists a quasi-commuted coalgebra structure on \( B(A) \) for each quasi-Hopf algebra \( A \) such that \( \alpha(A) : \Omega B(A) \rightarrow A \) is a morphism in \( \text{Alg}^*(R) \) and \( B : \text{Alg}^*(R) \rightarrow \text{Coalg}^*(R) \) is a functor.

**Proof.** We define \( \Delta^* \Omega B(A) : \Omega B(A) \rightarrow \Omega B(A) \otimes \Omega B(A) \) as the composite
\[
\begin{array}{c}
\Omega B(A) \\
\xrightarrow{\Omega B \Delta^*(A)} \\
\Omega B(A \otimes A) \\
\xrightarrow{\psi(A, A)} \\
\Omega B(A) \otimes \Omega B(A).
\end{array}
\]
The desired properties of \( \Omega B \Delta^*(A) \) now follow from those of \( \psi(A, A) \) considered in 5.5. This proves the theorem. \( \square \)

The next proposition will be used in the following section to show that certain morphisms \( C \rightarrow B(A) \) from a commutative coalgebra \( C \) are morphisms of quasi-commuted coalgebras.
5.7. Proposition. Let
\[ \sigma : B(A \otimes A') \to B[\Omega B(A) \otimes \Omega B(A')] \]
be the morphism adjoint to
\[ \psi : \Omega B(A \otimes A') \to \Omega B(A) \otimes \Omega B(A'). \]
Then the following diagram is commutative:

\[
\begin{array}{ccc}
B(A) \otimes B(A') & \xrightarrow{\gamma} & B(A \otimes A') \\
\beta \otimes \beta & \downarrow & \downarrow \sigma \\
B \Omega B(A) \otimes B \Omega B(A') & \xrightarrow{\gamma} & B[\Omega B(A) \otimes \Omega B(A')] \\
\end{array}
\]

Proof. With the notation of 5.5 we have a retraction
\[ r : B[\Omega B(A) \otimes \Omega B(A')] \to BN(A, A') \]
and the existence of \( \sigma \) with \( B(\alpha \otimes \alpha) = B(A \otimes A') \) is equivalent to the statement that
\[ \theta = B(\alpha \otimes \alpha) \circ r : B[\Omega B(A) \otimes \Omega B(A')] \to B(A \otimes A') \]
is an isomorphism. Hence \( \sigma \gamma = \gamma(\beta \otimes \beta) \) if and only if
\[ \text{pr}_i \theta \sigma \gamma = \text{pr}_i \theta \gamma(\beta \otimes \beta) \quad \text{for } i = 1, 2, \]
where \( \text{pr}_i \) are the projections onto the two factors of \( B(A \otimes A') \otimes BN(A, A') \).

For \( i = 1 \), note that \( \text{pr}_1 \theta = B(\alpha \otimes \alpha) \). First,
\[ B(\alpha \otimes \alpha) \gamma(\beta \otimes \beta) = \gamma(B(\alpha \otimes \alpha) \beta \otimes \beta) = \gamma((B(\alpha) \beta \otimes (B(\alpha) \beta) = \gamma(B(A) \otimes B(A'))) \gamma = \gamma. \]

For \( i = 2 \), note that \( \text{pr}_2 \theta = r \) and \( r \sigma \gamma = \ast \gamma = \ast \). Thus it remains to show that
\[ r \gamma(\beta \otimes \beta) = \ast, \]
where \( \ast = e \eta \) as usual. Recall from (5.5) that
\[ IN(A, A') = IN(A) \otimes IN(A') \oplus [I(A) \otimes IN(A')] \oplus [IN(A) \otimes I(A')] \oplus [IN(A) \otimes I(A')] \]
From the splitting lemma argument 2.5 the retraction \( r = h^{-1} w \), where \( h \) is defined by the commutativity of the diagram

\[
\begin{array}{ccc}
BN(A, A') & \xrightarrow{\text{inclusion}} & B[\Omega B(A) \otimes \Omega B(A')] \\
& & \downarrow w \\
& & T(IN(A, A')) \\
\end{array}
\]
Moreover, \( r \gamma(\beta \otimes \beta) = \ast \) if and only if \( w \gamma(\beta \otimes \beta) = \ast \). This holds if and only if
\[ \tilde{w} \tilde{l}(\gamma(\beta \otimes \beta)) = \ast \] in the commutative diagram
where $p$ is a suspension with a projection. Since the image into $I(\Omega B(A) \otimes \Omega B(A'))$ from $I(B(A) \otimes B(A'))$ is in the factor $I(A \otimes A')$, its projection to $sI(N(A, A'))$ is zero. Thus $\bar{w}(\gamma(\beta \otimes \beta)) = *$ and $\sigma \gamma = *$, which in turn proves that $\gamma = \gamma(\beta \otimes \beta)$. □

6. Study of the coalgebra $BC_{*}(T)$ for a torus $T$

We are interested in lifting the commutative coalgebra $H_{*}(BC_{*}(T))$ into $BC_{*}(T)$ for a torus $T$ with the following general proposition which uses the notation of divided powers which is discussed in [9].

6.1. Proposition. Let $A$ be a commutative Hopf algebra with primitive elements $v_i$ in odd degrees for $1 \leq i \leq m$, and let $S'(s(v_1), ..., s(v_m))$ be the free commutative coalgebra with its additional structure as a Hopf algebra with divided powers. There is a unique morphism

$$f : S'(s(v_1), ..., s(v_m)) \rightarrow H(A)$$

of algebras with divided powers such that $f(s(v_i)) = [v_i]$. Moreover, $f$ is a morphism of quasi-commuted coalgebras.

Proof. The existence and uniqueness of $f$ as a morphism of algebras with divided powers is immediate from the universal character of $S'$. For the last statement, observe first that the following diagram is commutative:

$$\begin{array}{ccc}
C = S'(s(v_1), ..., s(v_m)) & \xrightarrow{\Delta(C)} & B(A) \\
\downarrow & & \downarrow B(\Delta(A)) \\
C \otimes C & \xrightarrow{\Delta \otimes \Delta} & B(A) \otimes B(A) & \xrightarrow{\gamma} & B(A \otimes A)
\end{array}$$

Since this is a diagram of algebras with divided powers, it suffices to check commutativity on the elements $s(v_i)$. 
Calculate
\[
B(\Delta(A)) f(s(v_i)) = B(\Delta(A))[v_i] = [v_i \otimes 1] + [1 \otimes v_i]
\]
\[
\gamma(f \otimes f) \Delta(s(v_i)) = \gamma(f \otimes f)(s(v_i) \otimes 1 + 1 \otimes s(v_i))
\]
\[
= \gamma([v_i] \otimes 1 + 1 \otimes [v_i])
\]
\[
= [v_i \otimes 1] + [1 \otimes v_i].
\]

This establishes the commutativity of the diagram and also demonstrates that \(f\) preserves the coalgebra structure, because \(f(s(v_i))\) is primitive and \(s(v_1), \ldots, s(v_m)\) generate \(C = S'(s(v_1), \ldots, s(v_m))\) as an algebra with divided powers.

To see that \(f : C \to B(A)\) is a morphism of quasi-commuted coalgebras, we use the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
C \\
\Delta(C)
\end{array} \\
\downarrow
\begin{array}{c}
C \otimes C \\
f \otimes f
\end{array}
\end{array}
\begin{array}{c}
\rightarrow
\begin{array}{c}
B(A) \otimes B(A) \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\downarrow
\begin{array}{c}
\beta \otimes \beta
\end{array}
\end{array}
\begin{array}{c}
B \Omega B(A) \otimes B \Omega B(A) \\
\gamma
\end{array}
\begin{array}{c}
B(\Omega B(A) \otimes \Omega B(A))
\end{array}
\begin{array}{c}
\downarrow
\begin{array}{c}
B(\Omega f \otimes \Omega f)
\end{array}
\end{array}
\begin{array}{c}
\rightarrow
\begin{array}{c}
B(\Omega C \otimes \Omega C)
\end{array}
\end{array}
\end{array}
\]

The commutativity of the above diagram follows from the naturality of \(\beta\) and of \(\gamma\) and Proposition 5.7. This proves the proposition. \(\square\)

A torus \(T\) is a \(K(\pi, 1)\), where \(\pi\) is a free abelian group, and we can use the canonical simplicial model. For the commutative Hopf algebra \(C_*(T)\) we form \(BC_*(T)\). The double homology suspension \(\sigma \sigma : \pi \otimes R \to BC_*(T)_2\) defines a natural homomorphism \(H_2(BC_*(T)) \to BC_*(T)_2\), which induces the identity on \(H_2(BC_*(T))\) and a basis of \(H_2(BC_*(T))\) can be represented by primitive chains of \(C_1(T)\). Hence 6.1 applies, and we have the following proposition.

6.2. Proposition. The morphism \(H_2(BC_*(T)) \to BC_*(T)\) extends to \(u : H_*(BC(T)) \to BC_*(T)\) a morphism of Hopf algebras with divided powers such that \(H_*(u)\) is the identity and \(u\) is a morphism of quasi-commuted coalgebras.
7. Homological properties of a subtorus of a compact Lie group

7.1. Definition. A compact Lie group \( G \) is of exterior type over \( R \) provided
\[
H^*_G(G, R) = E(x_1, ..., x_n),
\]
an exterior Hopf algebra on generators \( x_1, ..., x_n \). It is of strictly exterior type provided all \( x_1, ..., x_n \) are odd dimensional.

If \( 2 \) is not a zero divisor in \( R \), then exterior type implies strictly exterior type. For characterizations of this concept see 8.1 in the next section.

Let \( D \) be a quasi-commuted coalgebra with a comultiplication \( \Delta \Omega(D) \) on \( \Omega(D) \), and let \( v_i : \Omega(D) \to A(i) \) for \( 1 \leq i \leq n \) be morphisms of algebras. By choosing a particular iterate \( \Delta_n \Omega(D) : \Omega(D) \to \Omega(D)^{n \otimes} \) and composing with \( v_1 \otimes ... \otimes v_n \), we have a morphism of algebras
\[
\bar{v} = (v_1 \otimes ... \otimes v_n) \Delta_n \Omega(D) : \Omega(D) \to A(1) \otimes ... \otimes A(n).
\]

Let
\[
v : D \to B(A(1) \otimes ... \otimes A(n))
\]
denote the adjoint of the assembled morphism \( \bar{v} \).

7.2. Proposition. Let \( f : C \to D \) be a morphism of quasi-commuted coalgebras, where \( C \) is commutative with its natural quasi-commuted structure, and let
\[
v : D \to B(A(1) \otimes ... \otimes A(n))
\]
be the adjoint to the assembled morphism
\[
\bar{v} = (v_1 \otimes ... \otimes v_n) \Delta_n \Omega(D).
\]
where \( v_i : \Omega(D) \to A(i) \) are algebra morphisms. Then there exists a morphism
\[
h : C \to B(A(1)) \otimes ... \otimes B(A(n))
\]
of coalgebras such that the following diagram is commutative:

\[
\begin{array}{c}
C \xrightarrow{f} D \\
h \downarrow \quad \downarrow v \\
B(A(1)) \otimes ... \otimes B(A(n)) \xrightarrow{\gamma} B(A(1) \otimes ... \otimes A(n))
\end{array}
\]

Proof. The following diagram is commutative from the definition of the quasi-commuted structure on \( C \) and the fact that \( f \) is a morphism of quasi-commuted coalgebras:
The large rectangle is commutative by definition of \( f \) being a morphism of quasi-commuted coalgebras, and the other rectangles are commutative by the naturality of \( \gamma \). For
\[
h = \left[ \bigotimes_i B(u_i) \right] (B\Omega)^{n\otimes} \beta(C)^{n\otimes} \Delta_n(C)
\]
we deduce the desired commutative diagram. \( \square \)

Now we use 6.2 and 7.2 to prove the following basic proposition, which will, in turn, easily imply the collapsing theorem.

**7.3. Proposition.** Let \( T \) be a subtorus of a compact Lie group \( G \) of strictly exterior type over \( R \) with \( H_\bullet(G) = E(x_1, \ldots, x_n) \). We have morphisms of coalgebras
\[
u : H_\bullet(BC_\bullet(T)) \to BC_\bullet(T), \quad g : BC_\bullet(G) \to BE(x_1, \ldots, x_n)
\]
such that \( H_\bullet(\nu) \) and \( H_\bullet(g) \) are isomorphisms, and the threefold composite is factored by the inclusion \( S'(s(x_1), \ldots, s(x_n)) = BE(x_1, \ldots, x_n) \):
\[
H_\bullet(BC_\bullet(T)) \xrightarrow{\nu} BC_\bullet(T) \xrightarrow{i} BC_\bullet(G) \xrightarrow{g} BE(x_1, \ldots, x_n)
\]
\[
S'(s(x_1), \ldots, s(x_n)) = BE(x_1) \otimes \cdots \otimes BE(x_n)
\]

**Proof.** The morphism \( \nu : H_\bullet(BC_\bullet(T)) \to BC_\bullet(T) \) is given by 6.2. The induced morphism \( i : BC_\bullet(T) \to BC_\bullet(G) \) is a morphism of quasi-commuted coalgebras by 5.6.
Since
\[
H_\bullet(BC_\bullet(G)) = S'(\sigma(x_1), \ldots, \sigma(x_n)),
\]
we have morphisms of chain complexes $BC_\bullet(G) \to R \cdot \sigma(x_i)$ which lift to a morphism of coalgebras

$$BC_\bullet(G) \to T'(\sigma(x_i)) = S'(\sigma(x_i)) = B(E(x_i)).$$

The adjoint algebra morphism $v_i : \Omega(BC_\bullet(G)) \to E(x_i)$ assembles to

$$v = g : BC_\bullet(G) \to BE(x_1, ..., x_n)$$
as in 7.2 since $E(x_1) \otimes ... \otimes E(x_n) = E(x_1, ..., x_n)$. Clearly $H_\bullet(g)$ is an isomorphism from the definition of the $v_i$. Finally the factorization property follows from 7.2. This proves the proposition. ε

8. The collapsing theorem

8.1. Remark. The following statements are equivalent for a connected compact Lie group $G$ and a ring $R$:

1. the group $G$ is strictly of exterior type over $R$;
2. the homology $H_\bullet(G)$ over the integers has $p$-torsion only for primes $p$ which are units in $R$;
3. the $R$-module $H_\bullet(B(G), R)$ is free, and for a maximal torus $M$ of $G$ the co-module $H_\bullet(B(M), R)$ is an injective over $H_\bullet(B(G), R)$.

The equivalence of (1) and (2) is an easy application of the Bochstein spectral sequence, and the equivalence of (1) and (3) is given in [6, pp. 93–94].

8.2. Theorem. Let $G$ be a connected compact Lie group of strictly exterior type over $R$, and let $H$ be a connected closed subgroup of strictly exterior type over $R$. Then the induced fibre spectral sequence for the fibration $G/H \to B(H) \to B(G)$ collapses, where

$$E^2 = \text{Cotor}^{H_\bullet(B(G))}(R, H_\bullet(B(H))).$$

Proof. For a maximal torus $T$ of $H$, the $H_\bullet(B(H))$-comodule $H_\bullet(B(T))$ is injective by 8.1. From this and the factorization

$$H_\bullet(B(T)) \to H_\bullet(B(H)) \to H_\bullet(B(G))$$

we see that $E^2 = \text{Cotor}^{H_\bullet(B(G))}(R, H_\bullet(B(H)))$ is a direct factor of $E^2 = \text{Cotor}^{H_\bullet(B(G))}(R, H_\bullet(B(T)))$, and in fact the whole spectral sequence $E^r$ is a direct factor of $E^r$ for $r \geq 2$. Hence it suffices to prove the theorem for $T \to G$.

The induced fibre spectral sequence is defined algebraically from the morphism of coalgebras $C_\bullet(B(T)) \to C_\bullet(B(G))$ or equivalently $BG_\bullet(T) \to BC_\bullet(G)$. Using the factorization of 7.2, namely

$$H_\bullet(BC_\bullet(T)) \xrightarrow{u} BC_\bullet(T) \xrightarrow{g} BC_\bullet(G) \xrightarrow{S'} \to BE(x_1, ..., x_n)$$

we have
we deduce a sequence of isomorphisms of spectral sequences which we indicate for the differential Cotor functors:

\[
\begin{align*}
\cotor^B_{BC_\ast(G)}(R, BC_\ast(T)) \\
\uparrow \\
\cotor^B_{BC_\ast(G)}(R, H_\ast(BC_\ast(T))) \\
\downarrow \\
\cotor^B_{E(x_1, \ldots, x_n)}(R, H_\ast(BC_\ast(T))) \\
\uparrow \\
\cotor^S(s(x_1), \ldots, s(x_n))(R, H_\ast(BC_\ast(T)))
\end{align*}
\]

In the last Cotor both differential coalgebras involved have zero differential, and thus \(F^2 = F^{\infty}\) in this case. Hence the spectral sequence for the fibration collapses, an this proves the theorem. □

References